Math. lin 76

Jonas

Mathesis. Systemata & methodi. 110.
Synopsis Palmariorum Matheseos:
OR, A NEW INTRODUCTION TO THE MATHEMATICS:
Containing the PRINCIPLES OF Arithmetic & Geometry
Demonstrated, In a Short and Easie Method;
WITH Their Application to the most Useful Parts there-
of: As, Resolving of Equations, Infinite Series,
Making the Logarithms; Interest, Simple and Com-
pound; The Chief Properties of the Conic Sections;
Mensuration of Surfaces and Solids; The Fundamen-
tal Princets of Perspective; Trig-

Design'd for the Benefit, and adapted to the Capacities of BEGINNERS.

By W. JONES.

TO THE

HONOURABLE

William Lowndes, Esq;

SECRETARY

TO THE

TREASURY.

SIR,

In this Attempt, I fear, I have incurrd a double Censue:\nOne by Inscribing a Small Tract

A 2 of
DEDICATION.

of ARITHMETIC, to so Great a Master in Numbers; And another by throwing in my poor endeavors among the Public Affairs, and Crowd of Transactions of the Highest Concern, which Continually pass thro' Your Hands.

I shou'd, therefore, offer at an Apology for my presumption; And entreat Your favorable Interpretation of my performances.

I might inform You too, Sir, of the Forwardness of my Zeal to produce something more worthy Your Acceptance, by a future Progress in these Studies.

My Devotion would lead me farther: But remembering how Valuable every Moment must be to a Person in Your Station, and to One that so Faithfully, so Exactly discharges the Important Business of it; I only crave the Honor
DEDICATION.

Honor to Declare, how Ambitious I am of shewing my just Esteem and Respect, and how Proud to subscribe my self,

SIR,

Your most obedient,

most humble Servant,

W. Jones.
THE

PREFACE.

The greatest part of the following Sheets was drawn up, and put into this form, for the Use of some Friends, who had neither Leisure, Convenience, nor, perhaps, Patience to search into so many different Authors, and turn over so many tedious Volumes, as is unavoidably requir'd to make but a tolerable Progress in the Mathematics; And the Benefit receiv'd by the Method we have taken, encourag'd us to Publish the same, for the Use of Learners, whether inclin'd to the Study of them, or occasionally oblig'd to be Employ'd in the Practice of some parts of this Science; whose Principles, therefore, they are concern'd to be well acquainted with, but are unwilling to be incumber'd with more than is just necessary. The Particulars insisted upon in this Treatise are as follow.

1. The Fundamental Rules of Numeral Computation or Common Arithmetic in whole Numbers are briefly, but fully deliver'd, with the necessary Examples in several Cases; not omitting the most Simple Compendiums in the Operations, that might be of any Use to Practitioners.

2. The First Principles of Literal Computation, usually called Algebra, are illustrated with Variety of Instances, and the Reasons of the several Operations.

3. The usual Methods, as well of Raising Simple Powers, as of Extracting the Roots of such Powers, with Examples both in Numbers and Species.

4. The Nature and Properties of Proportion; where we have bin designedly large, adapting, as far as possible, everything to the Capacities of young Beginners, that they might not be to seek for their Chief Materials, in professing these Studies: For the whole Body of Mathematics is, in reality, nothing else, but the Doctrine of Proportion: since it only comprehends whatever admits of Greater or Less, as such.

5. Fra-
The Preface.

5. Fractions, Vulgar and Decimal; (being only Operations about Proportions) where we have bin no less particular, well knowing, that more than an Obseure Conception of these Things is required in him that deservers the Name of an Accompanist. And that the Reader might be compleetly furnish'd with what is necessary to the Understanding the following part of this Treatise, we have given Variety of Examples both Numerall and LITERAL in the severall Cases.

6. The Arithmetic of Surd or Incommensurable Roots is here made plain and easie: Being Operations frequently used in the Practice of Algebra.

7. Some general Directions are laid down for the Solution of Mathematical Problems; With the Methods of Reducing Equations, Exemplified in all the Cases; As also the Derivation, Composition, and Solution of Equations. And the Method used, as well in the few easy Instances given in Simple Equations, as in the Ordinary Rules of Common Arithmetic (which fall under that Head) if well understood and practis'd, is sufficient to enable the Reader for the Solution of any Question whatever relating to any of these Rules, and that after the most Compendious Manner.

8. Arithmetical Progression, with the Solution of the severall Cases, and the Operations at Length; that the Learner might be further acquainted with Analytical Investigations, and the Method of proceeding in the like Calculations. Together with some not unuseful Rules relating to the Arithmetic of Infinites; by which we have, in its proper place, demonstrated, briefly and directly, some of the most considerable Propositions in Geometry. Whence also we have explained the Nature and Properties of Figure Numbers; and shewn the Method of Summing up a Series of such Numbers. From these of course are deriv'd the General Theorems for Extracting the Root of any Binomial or Infinitesimal Power, or for Raising such a Power; On which we insist'd the more, because the Method of Approximations by Infinite Series depends thereon: Besides, many valuable Rules and Conivances for the more easy Solution of some of the greatest Difficulties in Mathematics are due as so many immediate Consequences drawn from hence; and, indeed most of what we call Late Improvements, or New Methods of Investigations, are little else, than the due Application of these Rules.
The Preface.

A Remarkable Instance of the Use of this Theorem is given, in the next place, for making those Numbers called Logarithms, with several Examples, and such Explications of the Nature of them, as may render their Construction and Use intelligible to ordinary Capacities. Other Eminent Theorems drawn from hence; are those for Extracting the Roots of Finite and Infinite Equations; The former we intend would to explain from the successive curious Improvements of Mr. Raphson, Mr. Halley, and Mr. Sharp, with such Illustrations and easy Directions, as might complete the Learner in the Management of Natural Numbers, by the most expedients and ready Methods.

9. Increasing and Decreasing Geometric Progressions, with the Solution of the several Cases therein, and the Operations at Length.

10. Interest, Simple and Compound, solved in all its Cases.


12. The Rudiments of Geometry, wherein the most valuable and useful Propositions in Euclid, Archimedes, Apollonius, and others, are demonstrated in a Natural, Concise, and Easy Method; The principal Properties of (those Figures called) the Conic-Sections are consider'd both with, and without respect to the Cone. Together with Rules for the Mensuration of Lengths, Surfaces, and Solids, investigated by the common general Methods; with the Application of some of those Rules to Calk-Giuging: In the Instances there given, some Quantities (as the Learned Dr. Wallis look'd upon 'em) are determin'd by Summing up their Elements, by the Arithmetic of Infinites; Some (as the Incomparable Sir I. Newton consider'd 'em) by the Velocities of the Motion or Increments, by which they are Generated; according as they render'd the matter either more short, easy, or general: And, for a further Illustration, some by both these Methods.

13. The Principles of Projection, containing the Fundamental Rules for the Practice of Perspective, or for representing any Object as it appears to the Eye in any given Situation: with the Laws of the Orthographic, Stereographic, and Gnomic Projection of the Sphere, by which any one may readily project the same upon the Plane of any great Circle, or the several Cases of Spheric Triangles, and measure any Arc or Angle when projected; As also Delineate any Distil, and with Ease describe the Parallels of the Sun's
The Preface.

Sun's Declination, or any other Furniture, on the Plane thereof.

14. Trigonometry both Plane and Spherical, with the Rules necessary for the Solution of any Case therein, briefly demonstrated.

15. The Principles of Mechanics, with the general Laws of Motion and their Application in explaining the Powers of Simple Machines and Engines; with some Select Theorems from Galileus, Sir H. Newton, and Hugeni.us, relating to the Motion of Pendulums, Centripetal Forces, Centres of Gravity, &c. To which is added the Doctrine of the Motion of Projects, particularly applied to Gunney and Throwing of Bombs; with Directions how to lay a Gun or Mortar to pass so as to strike a Mark with the greatest Certainty and Advantage: where we endeavor'd to follow the Steps of the Learned Geometer Mr. Halley. As also the Common Principles of Optics, wherein the chief Properties of Refracted and Reflected Rays are briefly delivered; with that useful and general Rule, given by the last mention'd Excellent Person, for the Principal Foci of Refracted, or Reflected Rays, on any Figure'd Surface.

The whole is perform'd with as much Perspicuity and Plainness, as the Subject and our Limits would admit; and tho' we have not bin over nice in ranging the Particulars of this Treatise, yet we carefully observ'd the Method used by the most Eminent Mathematicians, who in their Writings were pleased to condescend to the Capacities of Beginners, as more especially, in the Arithmetical parts, Vieta, Oughtred, Tacquet, and Wallis. Nor have we bin less wanting in consulting the Works of the most Celebrated Ancient and Modern Geometers, that thereby no considerable and useful Proposition or Observation already publish'd, might escape our Notice. We have every where endeavor'd to express things after the clearest and most intelligible manner; and at the same time have avoided all unnecessary Curiosities that might Cloy the Fancy, or Burden the Memory. We likewise shun'd the tedious Pomp of Linear Demonstrations, affected by some, in things purely Arithmetical,
PREFACE.

those, which depend upon Proportion and the Common Asse-
dations of Quantities in general, are reckon'd to be; ) since
Analytical Demonstrations are not only more General and
Abstract, and therefore more Universally applicable to Par-
ticular Occasions, but also more Plain and Simple and alto-
gether as Scientific, as those made by Lines and Figures: Nor
is there any Difficulty from such to disguise the matter so as
to make it look more Geometrical. And since the Excellency
of this Science, which requires Attention, is its being Plain
and plain; we therefore have in every thing Audia Bre-
vity with Perspicuity: So that we doubt not, but a Learner,
that does not want the necessary Qualifications of Diligence
and Industry, will find, with Advantage. This the whole is
(as design'd) A Compendium of the most Select and Pri-
mary Principles of Mathematics, and may serve, as Such,
as an useful Introduction to further Enquiries.

Thus having given our Impartial Reader a Brief Account
of this Treatise, we hope his Candor will prompt him favo-
rably to excuse what is amiss, and amend those Errors
which unavoidably attend Things of this Nature.

'Twould be here needless to expatiate on the Usefulness of
Mathematics, a Part of Humane Literature, to which
all the Concerns of Humane Life are deeply engaged; Our
Pleasure, Security, and Commerce are almost entirely pro-
cured, maintained and improved, by the Means of Civil,
Military, and Naval Architecture; and in these there
are Variety of illustrious Instances and surprizingly magni-
ificent Pieces of Art, wherein the Effects of Geometry ap-
ppear in an extraordinary manner. And since we are defini-
tive of Sense acute enough to discover the exact Bulk, Mo-
tion, and Figure of Bodies, on which their Properties depend,
being conceal'd from us, either by their Remoteness, Mi-
nuteness, or Transparency; Since therefore, this bounds our
ambitious Desires of viewing the more secret Works of Na-
ture, and our Endeavors to subject 'em to the Doctrine of
Magnitude and Numbers, whereby we might, beyond the
possibility of Opposition or Doubt, Accounts for the several
Phenomena contained in our Enquiries; Yet,
The Preface.

Est quodam prodire tenus, si non datur ultra;

and it's somewhat of Consent to the Inquisitive Mind (who
is ever the least satisfied with that, whose Cause is most
conceal'd) that by Mechanics and Optics, the Natural
Abilities are so far Assisted and Improv'd, as to dis-
cover immensely distant, or Extremely small Objects; Hence
we are put in a Capacity of making more Correct Esti-
mations, and of forming juster Notions of the Magni-
itude, Revolutions and Distances of those Suspicious
Fabrics of Nature, which are the constant Subject of
our Observations.
ERRATA.


The Reader is desir'd to correct these; and to excuse any other that possibly may have escap'd Notice.

Synopsis
SYNOPSIS
Palmariorum Matheseos.

General Definitions.

I.

The most part of the Objects of our Knowledge may be considered as Capable of Augmentation and Diminution; and our Idea of Things as far as they have that Capacity, is what we call Quantity: By which Word may be comprehended whatever can be properly said to have Parts.

SCHOLIUM I.

Under this Definition of Quantity, we may rank Extension, Number, Weight, Motion, Time, &c. The one being taken as Greater or Less, Heavier or Lighter, Swifter or Slower, &c. in relation to another of the same kind.

For Great and Small, &c. are only comparative Terms of Things that are Homogeneous.
General Definitions.

SCHOLIUM 2.

And since the Primary and most considerable Property of Quantity is a being Capable of More or Less; Therefore Quantities may be Added to, Subtracted from, Multiplied by one another, and Divided into the Parts they contain.

II.

The mutual Relation of two Things of the same kind Compared together, in respect of Quantity, is call'd Ratio; And the Similarity of Ratio's is call'd Proportion.

III.

The Knowledge of these Comparisons of Quantities, or the Relations they have one to another, is what is generally call'd Mathematicks.

IV.

All Quantities have their Parts either Continuous or Discrete, that is, either United or Separated.

And that Quantity, which has its Parts Separated, is call'd Multitude, and is the Subject of Arithmetick; But that Quantity, which has its Parts united, is called Magnitude, and is the Subject of Geometry.
General Definitions.

SCHOLIUM 1.

The word Part only denotes our manner of conceiving a Thing when consider'd in Relation to its Whole, and may be taken as an Indivisible (Component) of it; or as that whose further Divisibility is not then enquired into. Hence,

SCHOLIUM 2.

Unit is properly a Name given to any Quantity, consider'd as an Indivisible: and,

Number is a Collection of Units; when those Units are look'd upon as Whole the Number goes under the Name of an Integer; but if they are taken as Parts of a Whole, then it is called a Fraction.
The Explication of the Signs and Characters used in this Treatise.

\[\begin{array}{c|c}
\text{Signifies} & \text{Explanation} \\
\hline
\times & \text{Equality, or equal to.} \\
\div & \text{Majority, or greater than.} \\
\pm & \text{Minority, or less than.} \\
\sqrt[ ]{} & \text{More, or to be added.} \\
\sqrt[ ]{} & \text{Less, or to be subtracted.} \\
\hline
\end{array}\]

Other Signs or Abbreviations of Words that occur, are explain'd in their own Places.
SYNOPSIS
Palmariorum Matheos.
PART I.
Containing the
PRINCIPLES
OF
NUMERAL and LITERAL
ARITHMETICK.

SECTION I.
Of Numeral Integers.

CHAPTER I.
Notation of Numeral Integers.

DEFINITION I.

Numerical Notation teaches how to express in Characters any Number proposed in Words.

S C H O
## SCHOLIUM

1. The characters used to express Numbers by, are either

<table>
<thead>
<tr>
<th>Marks</th>
<th>Names of the Arabians</th>
<th>Or</th>
<th>Marks</th>
<th>Names of the Romans</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>One</td>
<td></td>
<td>I</td>
<td>One</td>
</tr>
<tr>
<td>2</td>
<td>Two</td>
<td></td>
<td>V</td>
<td>Five</td>
</tr>
<tr>
<td>3</td>
<td>Three</td>
<td></td>
<td>X</td>
<td>Ten</td>
</tr>
<tr>
<td>4</td>
<td>Four</td>
<td></td>
<td>L</td>
<td>Fifty</td>
</tr>
<tr>
<td>5</td>
<td>Five</td>
<td></td>
<td>C</td>
<td>Hundred</td>
</tr>
<tr>
<td>6</td>
<td>Six</td>
<td></td>
<td>D</td>
<td>Five Hundred</td>
</tr>
<tr>
<td>7</td>
<td>Seven</td>
<td></td>
<td>M</td>
<td>Thousand</td>
</tr>
<tr>
<td>8</td>
<td>Eight</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Nine</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>Nothing or a Cypher</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Each of which Figures, besides their own single Value, receives several Denominations, according to their Place and Order.

3. And a Number has so many Places, as there are Figures in it; as, 364,87 is a Number of Five Places.

4. The Order in whole Numbers, is from the Right to the Left.

The Value of Places increases in a Decuple Proportion; for every Place to the Left, is Ten times the Value of the next Place to the Right.

6. Each Place also has its Name; and those Names, for the more easie Reading of large Numbers, are distinguished by Periods, half Periods, &c.

For as a Place

So a half Period is \( \frac{1}{10} \) of a Thousand

And a Period is \( \frac{1}{1000000} \) of a Million

7. A Cypher is of its self insignificant; but by its Place alters the Value of the Subsequent Figure.

8. And
Chap. 1. Palmariorum Matheos.

8. And since the value of each place is ten times the value of the next before it, 'tis certain, that

<table>
<thead>
<tr>
<th>Place</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>2</td>
</tr>
<tr>
<td>2nd</td>
<td>20</td>
</tr>
<tr>
<td>3rd</td>
<td>200</td>
</tr>
<tr>
<td>4th</td>
<td>2000</td>
</tr>
</tbody>
</table>

The Order and Names of Periods, &c.

<table>
<thead>
<tr>
<th>Periods</th>
<th>Half Decimals</th>
<th>Degree</th>
<th>Places</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units</td>
<td></td>
<td></td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Millions</td>
<td></td>
<td></td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>Billions</td>
<td></td>
<td></td>
<td>15</td>
<td>3</td>
</tr>
<tr>
<td>Trillions</td>
<td></td>
<td></td>
<td>20</td>
<td>4</td>
</tr>
</tbody>
</table>

COROL.
COROLLARY.

Whence the Value of each Figure in any Rank of Numbers, how large soever, is readily found by the following

RULE.

Begin at Units, set a Point under the 7th Place, then, reckoning that as one, count forwards, and set another under the next 7th Place, so continue to the end.

Then the Point from Units stands under

As is evident from the following

EXAMPLE.

<table>
<thead>
<tr>
<th>Periods</th>
<th>Quadrillions</th>
<th>Trillions</th>
<th>Billions</th>
<th>Millions</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Half Per.</td>
<td>th un</td>
<td>th un</td>
<td>th un</td>
<td>th un</td>
<td>th un</td>
</tr>
<tr>
<td>Degrees</td>
<td>th un</td>
<td>th un</td>
<td>th un</td>
<td>th un</td>
<td>th un</td>
</tr>
<tr>
<td>Figures</td>
<td>1234567890</td>
<td>987654321</td>
<td>0123456789</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

By this means you have as clear a Notion of, and may as easily read a Number of 70 Places, as that of seven.
CHAP. II.
Addition of Integers.

Definition.
Addition is the Collection of several Numbers or Quantities into one Sum.

Problem 1.
To add Integers of like Name into one Sum.

Rule.
Place all Numbers of like kind under one another, Add up the Units, and if their Sum be under Ten, set that Sum underneath; if equal to or above Ten, or Tens, set a Cipher underneath; and for every Ten carry an Unit to the next Place, and so proceed.

Example 1.
To add $237 + 432 = 669$.

The manner of Operation:
- $2 + 3 = 7$ which set under Units
- $3 + 3 = 6$ which set under Tens
- $4 + 2 = 6$ which set under Hundreds

$669$
235
432.

For there are \( \frac{600}{60} \) or \( \frac{7}{600} \) in the given Numbers.

Therefore \( 667 = 667 \) must be their Sum.

**Example 2.**

To 6985
Add 7645
And 4310

Sum 18940

The manner of Operation.

0 and 5 and 5 = 10
3 and 1 and 4 and 8 = 14
1 and 3 and 6 and 9 = 19
1 and 4 and 7 and 6 = 18

6985
7645
4310

For there are \( \frac{10}{13} \) Units
18 \( \frac{2}{18} \) Tens
17 \( \frac{3}{17} \) Hundreds

Therefore 18940 must be the Number required,
equal to all those given.

Since those things which are equal amongst themselves,
are also equal to one another.
EXAMPLE III.

To \(38796\)
Add \(4638\)
-----
\(30000\)  \(14\)  \(13\)  \(43434\)
\(12000\)  \(120\)  \(1212\)
\(1300\)  \(1300\)  \(3\)  \(14\)
\(120\)  \(12000\)
\(14\)  \(30000\)  \(43434\)

43434  43434

PROBLEM II.

To Add Integers of different Names.

RULE.

Place them severally in the same Line, with the Sign \(+\) between them, or understood to be so.

EXAMPLES.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>To 40</td>
<td>8 b</td>
<td>16 s.</td>
</tr>
<tr>
<td>Add 6</td>
<td>35 m.</td>
<td>9 d.</td>
</tr>
<tr>
<td>Sum = 40 + 6</td>
<td>8 b + 35 m.</td>
<td>16 s. + 9 d.</td>
</tr>
<tr>
<td>i.e. 46.</td>
<td>8 b + 35 m.</td>
<td>16 s. + 9 d.</td>
</tr>
</tbody>
</table>
SYNOPSIS

PART I. SECT. I.

PROBLEM III.

To Add Integers and Parts.

EXAMPLE I.

To 5f. 16s. 7d. or 6f. 4s. 1d.
Add 6. 14. 3d. or 7. 2.
Sum 11. 30. = 13. 6.

EXAMPLE II.

To 1641. 13s. 6d. or 1641. 15s. 6d.
Add 79. 12. 9. or 79. 12. 9.
Sum 244. 8. 3. = 243. 27. 15.

CHAPTER III.

Subtraction of Integers.

DEFINITION.

Subtraction is the taking of one Number or Quantity from another, to find their Difference.

SCHOLIUM.

The greatest of the given Numbers is called the Minuend. And the Number found is called the Remainder or Difference.
PROBL. I.

To Subtract Integers of like Names, when the Superior Numbers are greater than, or equal to their Inferiors.

RULE.

1. Place the Subducend under the Minuend, and draw a Line under both.
2. Begin at the Right-hand, take the left from the greater, or Equals from Equals, and set the Difference of each Row underneath.

Example, In Integers alone.

\[
\begin{array}{c}
\text{Minuend} & 638 \\
\text{Subducend} & 243 \\
\hline
\text{Remainder} & 425
\end{array}
\]

The manner of Operation.

\[
\begin{align*}
3 \text{ from } 6 & \text{ and there remain } 3 \text{ which set below,} \\
2 \text{ from } 8 & \text{ which is } 6 \text{ in the number.} \\
\text{That is,} & \\
3 \text{ from } 30 & \text{ remainder is } 20 \\
2 \text{ from } 600 & \text{ } 400 \\
\text{Therefor,} & 213 \\
\text{For since the Whole is equal to the Sum of all its Parts,} & \\
\text{therefore the Subduction of all the Parts is the same with the} & \\
\text{Subduction of the whole.} &
\end{align*}
\]

Examples,
Examples, In Integers and Parts.

Minuend 27d. 43s. 36d.
Subducend 12. 31s. 24d.

\[
\begin{array}{c}
\text{Remainder} \\
15. 17. 12.
\end{array}
\]
\[
\begin{array}{c}
124. 10. 4.
\end{array}
\]

PROB. II.

To Subtract Integers of the same Name, when some of the Superior Numbers are less than their Inferior.

RULE.

1. Place your Numbers, and begin as before.
2. And according to their respective Value, take one of the next Denomination, out of which Subduct; and so the Remainder, add the Superior, setting their Sum underneath.
3. Then add what you took to the next place on the Left-hand; and so proceed by this, or the former Rule.

Example, In Integers alone.

\[
\begin{array}{c}
\text{From} 2537 \\
\text{Subd. 1648} \\
\text{Rem. 889}
\end{array}
\]

The manner of Operation.

\[
1 + 4 = 5 \quad \text{from} \quad 17 \quad \text{Rem.} \quad 8 \quad \text{which set below}
\]

\[
1 + 6 = 7 \quad \text{from} \quad 15 \quad \text{Rem.} \quad 29 \quad \text{which set below}
\]

That
That is,

\[
\begin{align*}
10 & \quad 8 \\
100 & \quad 600 \\
1000 & \quad 1000 \\
\hline 
\text{Theref.} & \quad 1648 \\
\end{align*}
\]

\[
\begin{align*}
8 & \quad 7 \\
30 & \quad 100 \\
500 & \quad 1000 \\
2000 & \quad 0000 \\
\hline 
\text{Rem.} & \quad 89 \\
\end{align*}
\]

For by saying 8 from 17, I add Ten to the Minuend, but I add also the same to the Subducend, by saying 1 and 4 = 5; therefore the Remainder must be the same.

The Operation also may be thus;

\[
\begin{align*}
8 & \quad 17 \\
\frac{4}{6} & \quad \frac{13}{1} \quad \text{Rem.} \quad \frac{9}{8} \quad \text{Units} \\
1 & \quad \frac{15}{1} \quad \frac{2}{1} \quad \text{Rem.} \quad \frac{8}{8} \quad \text{Tens} \\
\frac{0}{0} & \quad \text{Hundred} \\
\end{align*}
\]

That is,

\[
\begin{align*}
8 & \quad 7 \\
40 & \quad 100 \\
600 & \quad 1000 \\
1000 & \quad 1000 \\
\hline 
\text{Theref.} & \quad 1648 \\
\end{align*}
\]

\[
\begin{align*}
80 & \quad 80 \\
800 & \quad 8000 \\
\hline 
\text{Rem.} & \quad 889 \\
\end{align*}
\]

For by adding a Ten to the Units, and taking it away from the Tens, the value of the Number is not changed.

Examples, In Integers and Parls.

From 5 s. 3 d. 7 i. e. 3 s. 15 d. or 4 s. 15 d.
Subd. 2. 9. 3. 9. 2. 9.
Rem. 2. 6. = 2. 6. = 2. 6.

From
THEOREM.

In Subtraction, the Subducend together with the Remainder, is equal to the Minuend.

For all the Parts taken together, are equal to the whole.

And if the Subducend be taken from the Minuend, there rests the Remainder.

But if a Part be taken from the Whole, the Remainder will be the other Part.

Therefore the Subducend, together with the Remainder are all the Parts of the Minuend, and consequently equal to it.

COROLLARY.

Hence, Addition and Subduction, serve Reciprocally to prove each other.

For Addition and Subduction are opposite in all Cases; and what is done by the one, is undone by the other.

Thus
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Thus if 6 be added 4, Sum 10
And if from 10 Subduct. 4
Rem. 6
That is, if 6 + 4 = 10, then 10 - 4 = 6.

CHAP. IV.
Multiplication of Integers.

Definition.

Multiplication is a manifold Addition, or the repeating a given Quantity, as often as required; That is, to take a Quantity so many Times, Part or Parts of a Time, as is represented by another.

SCHOLIUM.
The Number to be Repeated is called Multiplicand of Repetitions the Multiplier.
The Sum of the Number so often Repeated, is called the Product. Both Multiplicand and Multiplier are call’d Factors.

COROLLARY.
Hence, As Unit, is to one Factor:
So is the other Factor, to the Product.

D CASE
**CASE I.**

To Multiply single Numbers by one another.

**EXAMPLE.**

<table>
<thead>
<tr>
<th>Mult.</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>by</td>
<td>3</td>
</tr>
<tr>
<td>Prod.</td>
<td>12</td>
</tr>
</tbody>
</table>

i.e. \(4 + 4 + 4 = 4 \times 3 = 3 \times 4\).

**SCHOLIUM.**

All the variety that can happen in this Case, is expressed in the following

**Table of Multiplication.**

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
<td>9</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>6</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>21</td>
<td>28</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>22</td>
<td>32</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>9</td>
<td>18</td>
<td>27</td>
<td>36</td>
<td>11</td>
<td>11</td>
<td>11</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>
Chap. 4.  Palmariorum Matheseos.

LEMMA I.

The Product of any two Numbers, is equal to the several Products made by multiplying one of those Numbers by the several Parts of the other.

Thus \[ \frac{9}{3} \times \frac{3}{4} + 3 + \frac{2}{3} \times \frac{3}{2} = \frac{30}{2} + 4 \]

\[ 27 = 12 + 9 + 6 \]
\[ 68 = 60 + 8 \]

For, since the Whole, and all its Parts taken together, make but one and the same thing:

Therefore the Multiplying the Whole, and all its Parts, gives the same Product.

CASE II.

To Multiply a Compound Number by a Single one;

EXAMPLE.

Multiply \[ \frac{6452}{3} \times \frac{6000 + 400 + 50 + 2}{3} \]

Product. \[ 19356 = \frac{18000 + 1200 + 150 + 6}{3} \]

That is \[ \frac{6452}{3} \times \frac{6452}{3} + \frac{6452}{3} \times \frac{6452}{3} \]

\[ \begin{array}{c|ccc}
11156 & 6 & 18000 \\
82 & 150 & 1200 \\
19356 & 150 & 6 \\
\end{array} \]

\[ \begin{array}{c|c}
19356 & 19356 \\
\end{array} \]
20 Synopse Part 1, Sect. I.

For since $3 \times \left\{ \begin{array}{c} 6000 \\ 400 \\ 50 \\ 2 \end{array} \right\} = \left\{ \begin{array}{c} 18000 \\ 1200 \\ 150 \\ 6 \end{array} \right\}$

Therefore $3 \times 6452 = 19356$

**Lemma II.**

The product of any two quantities, is equal to the several products made by multiplying all the parts of one by all the parts of the other.

\[
\begin{array}{c}
\frac{12}{7} \times \frac{2}{3} \\
\frac{10 + 2}{3 + 4} \\
\frac{6 + 4 + 2}{3 + 4}
\end{array}
\]

\[
84 = 30 + 6 + 40 + 8 = 18 + 12 + 6 + 24 + 16 + 8
\]

**Case III.**

To multiply one Compound Number by another,

**Rule**

1. Place each number respectively under its kind.
2. Multiply each figure of the multiplicand, by each figure of the multiplier, and observe to set the first figure of each respective product under that figure of the multiplier by which it was made.
3. Add the several products together for the whole product.

**Example I.**

Mult. 123
by 23

\[
\begin{array}{c}
369 \\
246 \\
\hline
2829
\end{array}
\]

\[
\begin{array}{c}
\frac{123}{20} + \frac{123}{3} \\
\frac{100 + 20 + 3}{20 + 3}
\end{array}
\]

\[
2460 + 369 = 2000 + 400 + 60 + 300 + 60 + 9
\]

For
Chap. 4.  

\[
\begin{align*}
3 \times 100 & = 300 \\
20 & = 60 \\
3 & = 9 \\
100 \times 3 & = 2000 \\
20 & = 400 \\
3 & = 60
\end{align*}
\]

Therefore \(23 \times 123 = 2829\)

**Example 2.**

Multiplicand \(= 5326\)

Multiplier \(= 427\)

\[
\begin{align*}
37282 \\
10652 \\
21304
\end{align*}
\]

Product \(= 2274202\)

The manner of Operation.

I. 7 times \(\begin{align*}
2 & = 18 \\
3 & = 22 \\
5 & = 37
\end{align*}\)

\[
\begin{align*}
& 42 \text{ carry 5} \\
& 37
\end{align*}
\]

II. 2 times \(\begin{align*}
2 & = 12 \\
3 & = 6 \\
5 & = 10
\end{align*}\)

\[
\begin{align*}
& 5 \text{ carry 1} \\
& 5
\end{align*}
\]

III. 4 times \(\begin{align*}
2 & = 24 \\
3 & = 10
\end{align*}\)

\[
\begin{align*}
& 4 \text{ carry 2} \\
& 1
\end{align*}
\]
Therefore 2274202 is the Sum of the Products of all the Parts of the Multiplicand 5326, Multiplied by all the Parts of the Multiplier 427, and consequently equal to 5326 \times 427, by Lemma 2.

**S C H O L I U M I.**

When either the Multiplicand, Multiplier, or both, have Cyphers towards the Right-hand, then Multiply the Significant Figures by the former Rules, and annex to the Product as many Cyphers as there are in the Multiplicand and Multiplier.

**E X A M P L E S.**

246 by \[
\begin{array}{c}
10 \\
100 \\
1000 \\
10000
\end{array}
\]
gives \[
\begin{array}{c}
2460 \\
24600 \\
2460000
\end{array}
\]
SCHOLIUM 2.

When there are Cyphers in the Multiplier; then Substitute those Cyphers in Order, before the particular Product of the next Multiplier by the Multiplicand.

EXAMPLE.

\[
\begin{array}{ccc}
427 & 6700042 \\
306 & 100005 \\
\hline
2562 & 33500210 \\
12810 & 67000420000 \\
\hline
130663 & 67037700210 \\
\end{array}
\]

SCHOLIUM 3.

To Multiply by any Compound Number under 20; you need only,

Set the Product made by the Unis Figure of the Multiplier, a Place further to the Right-hand, and add thereto the Multiplicand. Thus,
Synopse Part. i. Sect. P.

Multiplicand 25436
Multiplier 14

\[ \text{Product} \ 356104 \]

Or shorter, thus.

25436
14

\[ \text{Product} \ 356104 \]

The manner of Operation.

\[
\begin{align*}
6 &= 24 \\
3 &= 12, \quad 2 + 6 &= 20 \\
4 &= 16, \quad 2 + 3 &= 21 \\
5 &= 20, \quad 2 + 4 &= 26 \\
2 &= 8, \quad 2 + 5 &= 15 \\
\text{Then} \quad 2 + 1 &= 3
\end{align*}
\]

Subt. 0

4

\begin{align*}
0 &\quad \times 2 \\
1 &\quad \times 2 \\
2 &\quad \times 2 \\
6 &\quad \times 2 \\
5 &\quad \times 15 \\
3 &\quad \times 3 \\
\end{align*}

\[ \text{Carry} 2 \]

SCHOLIUM 4.

If a Quantity be Multiplied by the Component Parts of the Multiplier, the Product will be the same as if it had been Multiplied by the Multiplier itself.

Thus, 245 by 7, and the Product by 6, is the same as if 245 was Multiplied by 7 \times 6, that is, by 42.
CHAP. V.

Division of Integers.

Definition.

Division is a Manifold Subduction; or the taking of one Number or Quantity out of another, as often as possible.

As 6 Divided by 2 gives 3.

For \( 6 \div 2 = 3 \).

The Number \( \{ \) to be Divided \( \} \) is called \( \{ \) Dividend \( \) Divisor \( \) \).

And the Number of Times they contain each other, is called Quotient.

Corollary.

Hence, as the Divisor, is to the Dividend;
So is Unit, to the Quotient.

The Terms in Division are thus Placed.

Divisor Dividend Quotient
2) 6 \( \div 3 \) (3

Or:

Dividend 6 \( \div (3 \) Quotient.
Divisor 2

PROB.
PROBLEM.

To Divide one Number by another.

RULE.

1. Set a Point under the left of the Left-hand Places in the Dividend, but of which the Divisor may be taken; and the Number of Places to the right of that Point inclusive, gives the Number of Places in the Quotient.

2. Try how often you can take the Divisor out of the first part of the Dividend, setting the Number of Times in the Quotient; then Multiply the Divisor thereby, Subtract the Product out of the said part of the Dividend, and subtract the remainder.

3. To the Right of the Remainder, set the next Figure of the Dividend; from which take the Divisor as often as you can, setting the Number of Times in the Quotient, Multiply the Divisor thereby, Subtracting the Product as before; and in this manner the Operation must be repeated, till you come to the end.

EXAMPLE I.

\[
\begin{array}{ccc}
\text{Divisor} & \text{Dividend} & \text{Quotient} \\
8 & 642 & (324) \\
\end{array}
\]
For 2 in \( \frac{600}{40} \left( \frac{300}{120} \right) \) times

\[
\begin{array}{c}
8 \ell \\
4 \\
2 \ell
\end{array}
\]

Therefore 2) 648 (324 times)

**EXAMPLE II.**

3) 19358 \( (6452) \)

\[
\begin{array}{c}
18 \\
13 \\
12 \\
15 \\
15 \\
98 \\
9 \\
\hline
Rem. 2
\end{array}
\]

For 3 in \( \left\{ \begin{array}{c}
18000 \\
1200 \\
150 \\
8 \end{array} \right\} \frac{6000}{40} \left( \frac{400}{50} \right) \) times

\[
\begin{array}{c}
18000 \ell \\
1200 \ell \\
150 \ell \\
8 \ell
\end{array}
\]

Rem. 2

Therefore 3) 19358 \( (6452) \)
The manner of Operation.

1. Having placed the Numbers, and pointed them as the Rule Directs: and finding I can have 24 in 56 but 2 times, therefore I set 2 in the Quotients, the Divisor Multiplied thereby, gives 48, which Subducted from 56 leaves 8; to the Right of 8 I set the next Figure of the Dividend, 6.

2. I can have 24 in 86 thrice, therefore I set 3 in the Quotients; then thrice 24 is 72, which Subducted from 86 leaves 14; to the Right of 14, I set 4 the next and last Figure of the Dividend.

3. I can have 24 in 144, 6 times, therefore I set 6 in the Quotients; then 6 times 24 is 144, which Subducted from 144, leaves nothing. Hence, I conclude, That 24 is contained 236 times in 5664.
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The reason of this Operation is evident from the following Process.

I. 24 in) 5600 (200 times
   \[200 \times 24 = 4800\]
   \[800\]
   \[+ \quad 60\]
   \[5664\]

II. 24 in) 860 (30 times
   \[30 \times 24 = 720\]
   \[140\]
   \[+ \quad 4\]
   \[864\]

III. 24 in) 144 (6 times
   \[6 \times 24 = 144\]
   \[900\]

And since \(\begin{cases} 24 & 4800 \text{ (200)} \\ 24 & 720 \text{ (30)} \\ 24 & 144 \text{ (6)} \end{cases}\) times

Therefore 24) 5664 (236

S C H O L I U M i.

If the Divisor be greater than the Dividend; or if they be of different Names, then, The Dividend set above, and the Divisor below a Line, represent the Quotient.

Ex. I. 4 Divided by 9, the Quotient is \(\frac{4}{9}\)

II. 24 Feet Divided by 6 Yards.
   24 Feet
   The Quotient is \(\frac{4}{6}\) Yards.

S C H O-
SCHOLIUM. 2.

If the Divisor have Ciphers towards the Right Hand; then,
Cut off so many of the Right Hand Places of the Dividend as there are Ciphers in the Divisor, which annex to the Remainder, when the Operation is finished.

EXAMPLES.

I.  10) 26718  (267 1/2
II. 300) 69152  (231 1/2
III. 3200) 245726 (76 1/1/2

... 224

217

192

25

THEOREM.

The Divisor Multiplied by the Quotient is equal to the Dividend.

As suppose 15 was Divided by 3, the Quotient will be 5. But the Quotient 5 is so many Parts of the Dividend 15, as Units is of the Divisor 3, that is, the third Part, by Definit. of Division.

Therefore, If the Quotients be taken so many times as there are Units in the Divisor 3, the Sum will be equal to the Dividend 15.

And all the Parts (5 times 3) taken together are equal to the whole (15).

Therefore, The Product of the Divisor by the Quotients is equal to the Dividend.

COROL.
Hence, 'tis evident, That Multiplication and Division serve Reciprocally to prove each other.

For Multiplication and Division are two contrary Operations, and what is done by the one, is undone by the other.

Thus, if 5 Multiplied by 3, give 15.
Then 5 is contain'd 3 times in 15.

Therefore \( \frac{15}{3} \) (\( = \)) 5 and \( \frac{15}{5} \) (\( = \)) 3.
Also if 15 Divided by 3 give 5,
Then 5 taken 3 times is 15.
Therefore \( 5 \times 3 = 15 = 5 \times 5 \).

SECT.
S E C T. II.

Of Literal or Algebraic Integers.

CHAP. I.

Notation of Algebraic Integers.

Definition.

Any number or quantity whatsoever, whether known or unknown, may be universally express’d by notes, characters or letters of the alphabet, at pleasure: and this way of notation, applied to arithmetical operations, is what is commonly call’d algebraical or literal arithmetic.

Scholium.

1. In operations perform’d by literal arithmetic, all quantities, known or unknown, are represented by letters, with prefix’d marks or signs, whereby, (according to the nature of the proposition) they may be so order’d by addition, subduction, multiplication, division, &c. as if each particular part was actually known; so that the
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the Position and Relation of one Quantity to another is visible thro' the whole Course of the Process, and consequently that of the known to the unknown.

2. Quantities represented by \( \frac{\text{the same}}{\text{different}} \) Letters in any Operation, are supposed to be of \( \frac{\text{the same}}{\text{different}} \) Value.

3. And between Quantities of different values, there may be an Equality, as \( 6a = 2b \); 6 Feet = 2 Yards.

4. The number of Times any Quantity is taken, must be prefix'd to it, and is call'd a Coefficient, or Co-factor.

5. A Quantity without a \( \frac{\text{Coefficient}}{\text{Sign}} \) is supposed to have \( \frac{\text{Unit}}{\text{the Sign} \pm} \) prefix'd to it.

6. Simple \( \frac{\text{Quantities}}{\text{have but one Member.}} \)

Compound \( \frac{\text{those which}}{\text{are connected by plus and minus.}} \)

And since Addition, Subtraction, Multiplication and Division, are the Common Affections of all Quantities, therefore we shall in the next plate endeavour with all the Brevity and Plainness possible, to apply these Rules to Letters, as we have to Numbers.

AXIOMS.

1. If to or from equal Quantities, equal ones be Added or Subducted, their Sum or Remainder will be equal.

2. If equal Quantities be Multiplied or Divided by equal ones, their Products or Quotients will be equal.
CHAP. II.

Addition of Algebraic Integers.

CASE I.

When Quantities have the same Name, and the same Sign,

RULE.

Let the propos'd Quantities be Collected, and their respective Signs adjourn'd.

EXAMPLES.

\[
\begin{array}{c|c|c|c|c}
\text{To} & +a & +2b & -2n & y+y+2n-2 \\
\hline
\text{Add} & +a & +5b & -n & y+y+3n-5 \\
\hline
\text{Sum} & +a & +5b & -3n & 2y+y+5x-7 \\
\hline
\text{Or} & +2a & +7b & \text{ } & \text{ } \\
\end{array}
\]

For 'tis manifest from the Common way of Numbering, that \(2 + 5 = 7\), of any thing of like Name; as 2 Miles and 5 Miles are 7 Miles.

CASE II.

When Quantities have the same Name, but different Signs,

RULE.

Let them be Subducted from each other, and the Sign of the Greater adjourn'd to the Remainder.

EX-
EXAMPLES.

To $+3a$ $-3x$ $+4\frac{1}{2}$ $6a\frac{1}{2}-x+30$
Add $-2a$ $+2x$ $-4\frac{1}{2}$ $x-20-6a\frac{1}{2}$

Sum $+3a-2a$ $-3x+2x$ $+4\frac{1}{2}-4\frac{1}{2}$ $+10$
Or $+a$ $-x$ $0$

For to Add a Negative, is to take away a Positive; Therefore, to connect a Negative and a Positive, is to make the one and destroy the other.

Thus, If $a$ has 600 l. and owes 400 l. 'tis plain that the Sum, or his Worth is but 200 l.

And if $a$ has 600 l. and owes 900 l. then the Worth is $-300$ l. or 300 l. worse than nothing.

CASE III.

When Quantities are of different Names,

RULE.

Let them be set down in order, with their own Signs prefixed.

EXAMPLES.

To $a$ $5b$ $a+b$ $3a-2x-3x+5$
Add $c$ $4m$ $c-d$ $47+pp+3a+2x$

Sum $a+c$ $5b+4m$ $a+b+c-d$ $6a+53-3x+pp$

For 4 Miles and 5 Hours, make neither 9 Miles nor 9 Hours.
C H A P. III.

Subduction of Algebraic Integers.

Case 1. + from +; and − from −

Case 2. − from +; and + from −

Both perform'd by this

General RULE.

Let (or suppose) all the Signs of the Subducend be changed,
Then the Quantities Collected, (as in Addition) give the Remainder.

EXAMPLES In Case 1.

From + 3a + ax − 2b + 5a + 4b + 6
Subd. + 2a + ax − 2b + 6a + 3b + 9
Rem. − 3a − 2a ax − bx − 2b + 2b 5a + 7 − 63
Or + a 0 0

For to take away any Thing, is the same as to subjoin the Defect of that Thing.
Therefore, to Affirmation of any thing, Negative take away the Negation is to make it Affirmative.

EXAMPLES, In Case 2.

From + 3a − 3a + 6a + 3a − 5a + 7
Subd. − 3a + 2a − 2a − y a + a − 2
Rem. + 5a − 5a + 8a + y 2a + 6a + 9

For to take away the want of a Thing, is to Add that very Thing, by taking away the Negation of it.

But to take away the Being or Affirmation of a Thing, must necessarily produce the Want or Negation of that Thing.

S C H O.
SCHOLIUM.

In *Addition* and *Subtraction*, it is indifferent as to Order, how the several Quantities do stand, so that each has its own Sign.

For \( x - y + z = x + z - y = -y + z + x = x + y = n \)

Or \( 8 - 6 + 2 = 8 + 2 - 6 = 6 + 2 + 8 = 2 + 8 = 6 + 4 \)

CHAP. IV.

**Multiplication of Algebraic Integers.**

**Note.**

That in *Multiplication* \( \{ \text{Like} \} \) \( \{ \text{Unlike} \} \) Signs give \( \{ + \} \) \( \{ - \} \) in the Product.

<table>
<thead>
<tr>
<th>Multiplicand</th>
<th>Multiplier</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm )</td>
<td>( \pm )</td>
<td>( \pm )</td>
</tr>
</tbody>
</table>

That is \( \{ \pm \} \) into \( \{ \pm \} \) gives \( \{ \pm \} \).

For to Multiply an *Affirmative* by \( \{ \text{an Affirmative} \} \) \( \{ \text{a Negative} \} \) \( \{ \text{Affirmation} \} \) \( \{ \text{of it}\} \) by Def. of *Mult.*

Therefore \( \{ \pm \} \) into \( + \), gives \( \{ \pm \} \).

But to Multiply a *Negative* by a *Negative* Quantity, is so many times to deny that *Negation*.

And so many times to deny the *Negation* of a *Thing*, is so many times to *Affirm* that *Thing*.

Therefore --- into --- gives +.

PROB.
PROB. I.

To Multiply Simple Quantities.

RULE.

Join the Factors together, and prefix to them the Product of the Coefficients, if there be any, observing the foregoing Note.

EXAMPLES.

<table>
<thead>
<tr>
<th>Mult.</th>
<th>+a</th>
<th>-a</th>
<th>+a</th>
<th>-a</th>
<th>+a</th>
<th>-3ab</th>
</tr>
</thead>
<tbody>
<tr>
<td>By</td>
<td>+b</td>
<td>-b</td>
<td>+b</td>
<td>-b</td>
<td>+3</td>
<td>5</td>
</tr>
<tr>
<td>Prod.</td>
<td>+ab</td>
<td>-ab</td>
<td>-ab</td>
<td>+3a</td>
<td>-15ab</td>
<td></td>
</tr>
</tbody>
</table>

PROB. II.

To Multiply Compound Quantities,

RULE.

Each part of the Multiplier must be drawn into each part of the Multiplier; by Lem. 2. Chap. 1. § 1.

Examples, Of Compounds by Simples.

<table>
<thead>
<tr>
<th>Mult.</th>
<th>x+y</th>
<th>x-y</th>
<th>6a+b+m</th>
</tr>
</thead>
<tbody>
<tr>
<td>By</td>
<td>x</td>
<td>x</td>
<td>3-a-n+m</td>
</tr>
<tr>
<td>Prod.</td>
<td>x-a+n</td>
<td>x-a+ (x-y)</td>
<td>3-6-a-n+m</td>
</tr>
<tr>
<td></td>
<td>x-x+y</td>
<td></td>
<td>-y</td>
</tr>
</tbody>
</table>
Examples, Of Compounds by Compounds.

Mult. $a + c$
By $a - c$
________________________

$aa + ac$
$-ac - cc$
________________________

Prod. $aa - ac$

$2x - a + 3$

$x - 4$

$2x - 2a + 3x$

$-8x + 4a - 12$

$2x - x - 5x + 4a - 12$

Mult. $4a^3 + 3a^2 - 2a + 1$
By $a^2 - 5a + 6$
________________________

Prod. $4a^5 - 17a^4 + 7a^3 + 29a^2 - 17a + 6$

Mult. $ZZZ - 2nZZ - nnn$

By $ZZ + 2nZ + 4nn$

________________________

$Z^5 - 2nZ^4 - 4n^2Z^3 - 8n^3Z^2 - 2n^4Z - 4n^5$

$+ a + 2na + a^3 + 2na^3 + 4n^2a^2$

$- 2n^2a - 4n^2a - 8n^3a$

$- a - 4na - a^3 - 4n^2a^2$

$+ 4n^3a + 8n^2a + 8n^3a^2$

$- 4na^2 + 4n^2a + a^3$

$- 2na^2 - 2n^2a^3$

$+ a^3$

Pr. $Z^5 *$

$* - 9n^3ZZ - 2n^4Z - 4n^6$

$+ 2a^3 + 2na^3 + 2n^2a^3$

$+ 10n^2a - 3n^3a - 4n^4a$

$- 6n^2a - 4a^4 - 4n^4a$

$+ 2n^3a + 7n^3a^3$

$+ 4a^4$

$S C H O -$
Synopsis


**SCHOLIUM 1.**

Sometimes **Products** are express’d only by the Quantities to be Multiplied with the sign \( \times \) between them.

Thus, The Product of \( a + x \) by \( e + y \), is \( a + x \times e + y \). And the Product of \( a + x \) by \( m - x + y \), and that Product by \( e + z r \)

Is \( a + x \times m - x + y \times e + z r \).

**SCHOLIUM 2.**

The Quantity produc’d by the Multiplication of Two, Three, &c. Quantities, is laid to be of Two, Three, &c. Dimensions; and the Quantities thus Multiplied, are called Roots;

\[ \{ a, b \} \] is a Quantity of Two Dimensions, whose Roots are \( a, b \).

\[ \{ a, b, c \} \] is a Quantity of Three Dimensions, whose Roots are \( a, b, c \).

But if the Roots are the same, then the Quantities produced are usually call’d Powers, as

\[ \{ a \} \] is the 1st. Power of \( a \), and may \( a^1 \).

\[ \{ a, a \} \] is the 2nd. Power of \( a \), and may \( a^2 \).

\[ \{ a, a, a \} \] is the 3rd. Power of \( a \), be thus express’d \( a^3 \).
Chap. 5. Palmariorum Matheos. 41

CHAP. V.

Division of Algebraic Integers.

General RULE.

Set the Divisor under the Dividend, with a Line between them.

EXAMPLES.

\[
\begin{array}{c|c|c|c}
\text{Divide} & a & a + bc - dd & an - nx \\
\text{By} & e & x - y & a + u \\
\text{Quotient} & \frac{a}{e} & \frac{a + bc - dd}{x - y} & \frac{an - nx}{a + u}
\end{array}
\]

For by Common Division, in Dividing 6 by 2, 'tis the same thing whether 3, or \(\frac{6}{2}\) be the Quotient.

But sometimes this Mark \(\div\) is used as a sign of Division.

SCHOOL I.

If a Quantity is found to be a Common Multiplier in both, it may be expunged from both; Observing that

\[
\begin{array}{c|c|c}
\text{Dividend} & \(_+\_\) & \(_+\_\) \\
\text{Divisor} & \(_-\_) & \(_-\_) \\
\text{Quotient.} & \(_+\_\) & \(_+\_\)
\end{array}
\]

From the like Reason with that in Multiplication.
Examples.

Divide \( \frac{a}{a+e} \) by \( \frac{a}{a+e} \) and \( \frac{a}{a+e} \) divided by \( \frac{a}{a+e} \).

Quot. \( \frac{a}{a+e} \) divided by \( \frac{a}{a+e} \) and \( \frac{a}{a+e} \) divided by \( \frac{a}{a+e} \).

For \( e \times a = a + e \), and \( a \times c \times n = a + e + n \) by theorem.

Chap. 5. § 1.

Divide \( \frac{a}{m+n} \) by \( \frac{a}{m+n} \) and \( \frac{a}{m+n} \) divided by \( \frac{a}{m+n} \).

Scholium 2.

In reducing these quotients into lower terms, it happens frequently, when the dividend and divisor are multinomial quantities, that there may be a common multiplier of both, which does not immediately appear; yet is found (as in ordinary arithmetic) by this

Rule.

Find one member of the quotient, by which multiply the divisor, subtracting the product from the dividend, and by the remainder seek another member of the quotient, and proceed thus, till the operation be finished.

Examples.

\[ \frac{x}{3} \] \( x - 3 \) \( x^3 - 3^3 \) (as \( x^3 + x^2 + 3^2 \))

\( 0 + x^3 - 3^3 \)

\( 0 + x^2 - 3^2 \)

\( 0 + 0 \)
The manner of Operation.

1. Find what Quantity Multiplied by \( x \) (the first Member of the Divisor) will give \( x^3 \), (the first Member of the Dividend) and that must be \( x^3 \); whereby Multiply \( x - 9 \), the Product \( x! - x^3 \); Subducted from the Dividend, leaves \( x^2 \) \( = 9 \).

2. Seek what Quantity Multiplied by \( 9 \), gives \( x^2 \) \( = 9 \), that must be \( 9 \times 9 \); which (as before) Multiplied and Subducted leaves \( x^2 = 81 \).

3. In the same manner, seek the third Member of the Quotient, which Multiplied and Subduced, leaves 0.

\[
\begin{align*}
\frac{x^6 - 19x^4 - 124x^2 - 64}{x^4 + 8x^2 + 4} & = \frac{-8x^4}{8x^4 - 128x^2} \\
& = \frac{-4x^2}{4x^2 - 64} \\
& = 0
\end{align*}
\]

\[
\begin{align*}
x^6 + n^3x^4 - n^4x^2 - n^6 & = (x^4 + 2n^2x^2 + n^4 - 2y^2 + y^4 - 2n^2y^2 - y^2 - n^2y^4) \\
& = (n^4x^2 - n^6 + n^2y^2 - 2n^4y^2 - n^2y^4)
\end{align*}
\]

5 C H O.
If the Divisor be not an even Part of the Dividend, the Operation may either be terminated by annexing to the Quotient, the Remainder set over the Divisor, with a Line drawn between them; or else continued on in an Infinite Series.

**Examples.**

Divide \( \frac{9n^3y^4}{2ny} \) by \( 2ny \) \( (= \frac{9n^2y^3}{2} \) \( \Rightarrow \frac{8n^3y^6}{4n^2y^3} \) \( (= \frac{2ny^5}{a} \)

\[
2n) \quad 8n^6 + 4n^3a^2y^2 - 3n^5a^4(4n^6 + 2n^2a^4y^4 - \frac{3n^6a^6}{2}
\]

\[
-2n) \quad 10n^5 - 12n^3a^2 - 6a^2c^3 (-5n^4 + 6n^2a^3 + \frac{3a^2c^3}{12}
\]

\[
4a - ec) \quad adc (e + \frac{e^3}{a^2} + \frac{e^5}{a^4} + \frac{e^7}{a^6} + \frac{e^9}{a^8} + \frac{85c}{a^{10}} - ec^3
\]

\[
\begin{align*}
eg^3 & - \frac{e^5}{a^2} \\
\frac{e^5}{a^2} & - \frac{e^7}{a^4} \\
\frac{e^7}{a^4} & - \frac{e^9}{a^6} \\
\frac{e^9}{a^6} & - \frac{85c}{a^{10}}
\end{align*}
\]
Chap. 5. Palmariorum Matheseos.

\[
4a - 3x) aaaa( ax + \frac{3^2 x}{a} + \frac{3^4 x}{aaa} + \frac{3^6 x}{a^3} + \text{&c.}
\]

\[
\frac{3^2 x}{a} - \frac{3^4}{a}
\]

\[
\frac{3^4 x}{a^3} - \frac{3^6 x}{a^3}
\]

\[
\frac{3^6 x}{a^3} - \frac{3^8 x}{a^5}
\]

\[
1 - x) 1 (1 + x + xx + x^3 + x^4, \text{&c.}
\]

\[
x
\]

\[
x - xx
\]

\[
xx - xxx
\]

\[
x^4
\]

S C H O.
In Multiplication and Division, 'tis indifferent (as to Order) how the Quantities stand, so the Operations be Performed successively.

CHAP. VI.

Involution of Quantities.

Definition.

A Quantity Multiplied into its self any Number of Times is said to be Involved; the Products arising are called Powers, and the Quantity so Multiplied a Root.

Thus $x^a \times a = a^2$ or $a^2$,

&c.

Here $a$ is the Root or First Power; $a^2$, $a^3$, &c. The Second, Third, &c. Powers of it.

SCHOLIUM.

Some of these Powers have borrowed their Denominations from Local Extension.

For a Line having but one Dimension, viz. Length, drawn into it self, produces a Square-Plane.

And that Square having two Dimensions, viz. Length and Breadth, drawn into it self, produces a Cubed-Solid.
Chap. 6.  *Palmariorum Matheseos.* 47

This Cube has three Dimensions, *viz.* Length, Breadth, and Thickness; But the Nature and Property of Space admits of no other Extension.

Whence it follows, That the Root or First Power being taken as a Side, the Second Power will be a Square, the Third a Cube.

**L E M M A A.**

Any Quantity (as a Root) Divided into Parts at Pleasure, the Sum of the Product of those Parts drawn into themselves, and into each other any Number of Times, will be equal to that Quantity drawn into itself the like Number of Times.

Thus, If the Root \( r \) be made a Binomial \( a + e \).

Then, \( r^2 = a^2 + 2ae + e^2 \) \( \text{first} \) Power.

And \( r^3 = a^3 + 3a^2e + 3ae^2 + e^3 \) \( \text{second} \) Power.

For,

First Power \( = \frac{a + e}{a + e} = \text{Root} \)

\[ \frac{a^2 + ae}{ae + e^2} \]

Second Power \( = \frac{a^2 + 2ae + e^2}{e} = \text{Square} \)

\[ \frac{a^4 + 3a^3e + 3a^2e^2 + 3ae^3 + e^4}{e^3} \]

Third Power \( = \frac{a^4 + 3a^3e + 3a^2e^2 + 3ae^3 + e^4}{e^3} = \text{Cube} \)

The Method of Proceeding is the same in the Generating of Higher Powers, from any given Root, whether Binomial, or Multinomial.
SYNOPSIS

PART I. SECT. 2.

COROLLARY.

Now from the Nature of the Dimension of each Species, their Place and Value in Numbers are soon Discovered.

As if the Root \( \sqrt{24} = \frac{a + \varepsilon}{2} \)

Its Square is \( \begin{align*}
20 \times 20 + 2 \times 20 \times 4 + 4 \times 4
\end{align*} \)

\( \text{i.e.} \quad 400 + 160 + 16 \quad = 576 \)

Its Cube is \( \begin{align*}
20 \times 20 \times 20 + 3 \times 20 \times 20 \times 4 + 3 \times 20 \times 4 \times 4 + 4 \times 4 \times 4
\end{align*} \)

\( \text{i.e.} \quad 8000 + 4800 + 960 + 64 = 13824 \)

Also for the Square of a Binomial Number.

<table>
<thead>
<tr>
<th>Root</th>
<th>75 = 70 + 5 = a + \varepsilon</th>
</tr>
</thead>
<tbody>
<tr>
<td>49</td>
<td>4900</td>
</tr>
<tr>
<td>70</td>
<td>700</td>
</tr>
<tr>
<td>25</td>
<td>25</td>
</tr>
</tbody>
</table>

\( \text{Square} \quad 5625 = 5625 = 5625 = a^2 + 2ae + \varepsilon^2 \)

And for the Cube.

<table>
<thead>
<tr>
<th>Root</th>
<th>75 = 70 + 5 = a + \varepsilon</th>
</tr>
</thead>
<tbody>
<tr>
<td>343</td>
<td>343000</td>
</tr>
<tr>
<td>735</td>
<td>73500</td>
</tr>
<tr>
<td>525</td>
<td>5250</td>
</tr>
<tr>
<td>125</td>
<td>125</td>
</tr>
</tbody>
</table>

\( \text{Cube} \quad 421875 = 421875 = a^3 + 3a^2 \varepsilon + 3ae \varepsilon^2 + \varepsilon^3 \)
In other Roots consisting of more Figures, the like Process is to be used.

1. Taking the Left-hand Figure for (a), and the next to it for (e); then by the Theorem for the Power set all the Parts in their due Places, and their Sum is the Power of those two Figures.

2. These two Figures being taken for (a), the next for (e), and Proceeding as before, you'll have the Power of the three Figures; and soon, till you have compleated the Power of the whole Root. Thus,

For the Square or Second Power of a Multinomial Number.

<table>
<thead>
<tr>
<th>4</th>
<th>6</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>.</td>
<td>..</td>
<td>= 40000 \times 40000 = ee</td>
<td></td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>.</td>
<td>..</td>
<td>= 40000 \times 6000 \times 2 = 2ae</td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>.</td>
<td>..</td>
<td>= 6000 \times 6000 = ee</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2116</td>
<td>.</td>
<td>..</td>
<td>= 46000 \times 46000 = ee</td>
<td></td>
<td></td>
</tr>
<tr>
<td>276</td>
<td>.</td>
<td>..</td>
<td>= 46000 \times 300 \times 2 = 2ae</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>.</td>
<td>..</td>
<td>= 300 \times 300 = ee</td>
<td></td>
<td></td>
</tr>
<tr>
<td>214369</td>
<td>.</td>
<td>..</td>
<td>= 46300 \times 46300 = ee</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4630</td>
<td>.</td>
<td>..</td>
<td>= 46300 \times 50 \times 2 = 2ae</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>.</td>
<td>..</td>
<td>= 50 \times 50 = ee</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21483225</td>
<td>.</td>
<td>..</td>
<td>= 46350 \times 46350 = ee</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64890</td>
<td>.</td>
<td>..</td>
<td>= 46350 \times 7 \times 2 = 2ae</td>
<td></td>
<td></td>
</tr>
<tr>
<td>49</td>
<td>.</td>
<td>..</td>
<td>= 7 \times 7 = ee</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2148971449</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Square.</td>
</tr>
</tbody>
</table>

\[ H \] For
For the Cube, or Third Power of a Multinomial Number.

<table>
<thead>
<tr>
<th>4</th>
<th>6</th>
<th>3</th>
<th>Root</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>...</td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td>288</td>
<td>..</td>
<td>400</td>
<td>400</td>
</tr>
<tr>
<td>432</td>
<td>..</td>
<td>400</td>
<td>60</td>
</tr>
<tr>
<td>216</td>
<td>..</td>
<td>60</td>
<td>60</td>
</tr>
</tbody>
</table>

| 97336 | .. | 460 | 460 | 460 = a a a |
| 19044 | .. | 460 | 460 | 3 x 3 = 3 a a e |
| 1242 | .. | 460 | 3 | 3 x 3 = 3 a e e |
| 27 | . | 3 | 3 | 3 = e e e |

99252847 Cube

The like Method is to be observed in the Composition of other Powers, each according to the Nature of the Theorem thereto belonging.

**S C H O L I U M.**

This way of raising Powers from their Component Parts, rather leads to the Method of finding the Root of a given Power, than to find the Power of a given Root, which is soon done by the Continual Multiplication of that Root.
Evolution of Quantities.

Definition.

The Resolution of Powers into their Roots is called Evolution or the Analysis of Powers.

SCHOLIUM.

And since any Root may be considered, as consisting of two Parts, one of which is supposed already known, and the other, the unknown, is yet discoverable by means of Theorems rais'd by the Involution of the Binomial Root; in which 'tis very easy to discern how each part of the Root is concern'd in the Power, and consequently how much is already known, and what remains farther to be enquired for.

Whence, the Method of Extracting the Root of any Power, may without difficulty be perform'd by observing only the Constitution of Powers, which plainly directs what manner of Operation each requires.

Now, the first thing to be done, is to distinguish the given Number into several Parts, by Points set over such Places as the Index of the Power directs.

Viz. That of Squares, Cubes, &c. into Two's, Three's, &c. beginning always from the Place of Units, and so towards the Left-hand in Integers, towards the Right-hand in Decimals; and there will be as many Places in the Roots, as there are Points.
The Method of Extracting the Roots of the Second Power.

Literal Examples.

I. \( a^2 + 2ac + ec \ (a + e) \)
\[
\begin{array}{c}
2a + e \ \ 0 \\
2ae + ec \\
2ae + ec
\end{array}
\]

II. \( a^3 - 6an + 9n^2 + 2ax - 6nx + x^2 \ (a - 3n + x) \)
\[
\begin{array}{c}
2a - 3n \ \ 0 - 6an + 9n^2 \\
-6an + 9n^2
\end{array}
\]
\[
\begin{array}{c}
2a - 6n + x \ \ 0 + 2ax - 6nx + x^2 \\
+2ax - 6nx + x^2
\end{array}
\]

III. \( a^4 + 4a^3c + 6a^2c^2 + 4ac^3 + c^4 \ (a^2 + 2ae + e^2) \)
\[
\begin{array}{c}
2a^2 + 2ae \ \ 0 + 4a^3c + 6a^2c^2 \\
+4a^3c + 6a^2c^2
\end{array}
\]
\[
\begin{array}{c}
2a^2 + 4ac + e^2 \ \ 0 + 2a^2c + 4ac^3 + c^4 \\
+2a^2c + 4ac^3 + c^4
\end{array}
\]
Chap. 7.  *Palmariorum Matheseos.*  53

**Example.**

Theoretically thus.

<table>
<thead>
<tr>
<th>$ae$</th>
<th>$2ae + ee$</th>
<th>178929 (423)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ae$</td>
<td>$4 \times 4$</td>
<td>$16 \ldots$</td>
</tr>
<tr>
<td>$2ae$</td>
<td>$2 \times 4 = 8$</td>
<td>$189 \ldots (e = e)$</td>
</tr>
<tr>
<td>$ae$</td>
<td>$8 \times 2$</td>
<td>$16 \ldots$</td>
</tr>
<tr>
<td>$ee$</td>
<td>$2 \times 2$</td>
<td>$4 \ldots$</td>
</tr>
<tr>
<td>$2ae$</td>
<td>$84 \times 2 = 168$</td>
<td>$2529 (3 = e)$</td>
</tr>
<tr>
<td>$ae$</td>
<td>$84 \times 3$</td>
<td>$252 \ldots$</td>
</tr>
<tr>
<td>$ee$</td>
<td>$3 \times 3$</td>
<td>$9 \ldots$</td>
</tr>
</tbody>
</table>

Practically thus.

<table>
<thead>
<tr>
<th>178929 (423)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$82$</td>
</tr>
<tr>
<td>$843$</td>
</tr>
<tr>
<td>$0000$</td>
</tr>
</tbody>
</table>

The Method of Extracting the Roots of the Third Power.

**Literal Example.**

\[
a^3 + 3a^2e + 3ae^2 + e^3 = (a + e)^3
\]

\[
3a^2 + 3ae + e^2 + 3a^2e + 3ae^2 + e^3
\]

| 0 | 0 | 0 | 0 |

**Numerical**
Synopsis \ Part. 1. Sect. 2.

**NUMERAL EXAMPLE.**

Theorically thus.

\[
\begin{align*}
\alpha^3 + 3 \alpha^2 \epsilon + 3 \alpha \epsilon^2 + \epsilon^3 &= 75686967(423) \\
\alpha^3 &= 4 \times 4 \times 4 = 64 \\
3 \alpha^2 &= 3 \times 4 \times 4 = 48 \\
3 \alpha^2 \epsilon &= 3 \times 4 \times 4 \times 2 = 96 \\
3 \alpha \epsilon^2 &= 3 \times 4 \times 2 \times 2 = 48 \\
\epsilon^3 &= 2 \times 2 \times 2 = 8 \\
3 \alpha^2 \epsilon + 3 \alpha \epsilon^2 + \epsilon^3 &= 10088 \\
3 \alpha^2 &= 3 \times 42 \times 42 = 5292 \\
3 \alpha \epsilon^2 &= 3 \times 42 \times 42 \times 3 = 15876 \\
3 \epsilon^2 &= 3 \times 3 \times 3 = 27 \\
3 \alpha^2 \epsilon + 3 \alpha \epsilon^2 + \epsilon^3 &= 1598967 \\
&= 0000000
\end{align*}
\]

Practically thus.

\[
\begin{align*}
75686967(423) \\
48) 11686 \\
96 \\
48 \\
8 \\
5292) 1598967 \\
15876 \\
1134 \\
27 \\
&= 0000000
\end{align*}
\]

**NOTE.**

If the Number is not an exact Square, Cube, &c. Annex to the Remainder, Two’s, Three’s, &c. of Cyphers, so any desired Number of Decimal Places may be had in that Essay.
S E C T. III.

Of the Comparisons of Quantities.

C H A P. I.

Of Proportion.

DEFINITION I.

THE Relation of two Homogeneous Quantities one to another, may be considered, either,

1. By how much the one Exceeds the other, which is called their Difference.

Thus 5 exceeds 3 by the Difference 2.

2. Or what Part or Parts one is of another, which is call'd Ratio.

Thus, the Ratio of $\frac{6}{3}$ to 3 is $\frac{2}{3}$ or Double $\frac{1}{2}$ to 3 is $\frac{3}{4}$ or Subduple.

Note, That the Quantity Compared is called the Antecedent; and that to which it is Compared is called the Consequent.
DEFINITION II.

When two \(\text{\{Differences\}}\), \(\text{\{Ratio's\}}\) are equal, the Terms that compose them are said to be \(\text{\{Arithmetically\}}\), \(\text{\{Geometrically\}}\) Proportional.

SCHOLIUM I.

Suppose the Terms to be \(a\) and \(b\), their Difference \(d\).

If \(a\) be the least \(\text{\{Term\}}\), then \(\frac{a + d}{a - d} = b\).

Therefore in any Arithmetic Proportion when the Antecedens is \(\text{\{less\}}\) than the Consequent, the Terms may be express'd by \(a\) and \(\frac{a + d}{a - d}\).

SCHOLIUM II.

Suppose \(a\) and \(b\) to be the Terms of any Ratio; If \(a\) be the least Term

Put \(r = \frac{b}{a}\), then \(ar = b\) by Equal Multi.

But if \(b\) be the least Term.

Put \(r = \frac{a}{b}\), then \(br = a\) by Equal Multi.

And \(\frac{a}{r} = b\) by Equal Division.

Therefore
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Therefore, in any Geometric Proportion, when the Antecedent is \( \frac{a}{r} \) greater than the Consequent, the Terms may be express'd by \( \frac{a}{r} \) & \( \frac{a}{r} \).

Definition III.

Those Quantities whose Excess or Quotients are the same, are call'd Proportional.

Case I.

When of several Quantities the Difference of the 1st. and 2d. is the same with that of the 2d. and 3d. they are said to be in a Continued Arithmetic Proportion.

Thus, \( \{a, a + d, a + 2d, a + 3d, a + 4d, \ldots\} \) is a Series of Continued Arithmetic Proportional, whose Common Difference is \( d \).

And \( \{a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \frac{a}{r^4}, \ldots\} \) is a Series of Continued Geometric Proportional, whose Common Multiplier is \( \frac{r}{1} \), or whole Ratio is that of \( r \) to \( 1 \).

Note, That the Sign \( \ldots \) Signifies Continued Proportion.
CASE II.

When of several Quantities the \(\frac{a}{a+d}\) of the 1st. and 2d. is the same with that of the 3d. and 4th. (and not of the 2d and 3d.) they are said to be in a Discontinued \(\frac{a}{a+d}\) Proportion; such as,

1. \(\frac{a}{a+d}, \frac{a}{a+d}; \frac{e}{e+d}, \frac{e}{e+d}\)  
   For \(\frac{a}{a+d} - a = \frac{e}{e+d} - e = d\).

2. \(\frac{a}{ar}, \frac{a}{ar}; \frac{e}{er}, \frac{e}{er}\) for \(\frac{a}{ar} = \frac{e}{er} = \frac{1}{r}\)  
   \(\frac{ar}{a} = \frac{er}{e} = \frac{r}{1}\)

---

CHAPTER II.

The Chief Properties of Arithmetic Proportion.

THEOREM I.

In any Number of Continued Arithmetic Proportionals:

1. If the Number of Terms be even;  
The Sum of the Extremes, and that of every two equally distant from them, are equal.

2. If the Number of Terms be odd;  
Then those Sums are each equal to the Double of the Middle Term.

Thus
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Thus in, \(a, a+d, a+2d, a+3d, a+4d.\)

It is evident that \(a+a+4d=a+d+a+3d = 2a+4d.\)

**Corollary I.**

Therefore, if three quantities are in a continued arithmetic proportion;

The sum of the extremes is equal to the double of the middle term.

**Prob. I.**

To find any number (\(n\)) of arithmetical mean proportionals, between any two given quantities, \(a,\) and \(e.\)

Since, \(a, a+d, a+2d, a+3d, \&c. a + n + 1 \times d = e.\)

Therefore, \(d \times n+1 = e-a;\) and \(d = \frac{e-a}{n+1}.\)

Or putting \(x = \text{first mean} = \text{second term},\)

Then \(a, x, x+a, 2x-a, 3x-2a, \&c. x \times n+1 = na = e.\)

Therefore \(x \times n+1 = e + na,\) and \(x = \frac{e+na}{n+1}.\)

Now having either the difference, or the first mean, the rest are soon found.

**Prob. II.**

To find a 3rd, 4th, 5th, or nth arithmetic proportional to any two given quantities, \(a,\) and \(x.\)

It is evident \(a-n+i \times d,\) or \(x \times n - i - a \times n - 2\)

is the proportional required by the preceding problem.
Synopsis Part I. Sect. 3.

Corollary II.

If in a Rank of Continued Arithmetic Proportionals, there be taken any Series of Equidistant Terms, That Series will be also Proportional.

For if \( a, a+d, a+2d, a+3d, a+4d, a+5d, \ldots \) then

\[
\sum a, a+d, a+2d, a+3d, a+4d, a+5d, \ldots
\]

Theorem 2.

If 4 Quantities be Arithmetically Proportional; The Sum of the Extremes is equal to the Sum of the Means.

That is, if \( a, a+d, a+2d, a+3d, \) Continued. \( a, a+d; \ c, c+d, \) Discontinued.

'Tis plain

\[
\frac{a+a+3d}{a+c+d} = \frac{a+d+a+2d}{2a+3d} = \frac{a+c+d}{a+d+c}.
\]

Chapter III.

The Chief Properties of Geometric Proportion.

Theorem 1.

In any Number of Continued Geometric Proportionals:

1. If the Number of Terms be even; The Products of the Extremes, and that of every two Terms, equally distant from them are equal.

2. If the Number of Terms be odd; Then those Products which are each equal to the Square of the Middle Term.

Thus,
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Thus, in

\[ a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, \]

Tis evident that,

\[ a \times ar^6 = ar \times ar^5 = ar^2 \times ar^4 = ar^3 \times ar^3 = a^2 \times r^6. \]

**Corollary 1.**

Hence, if three quantities are in a Continual Geometrical Proportion;

The Product of the Extremes is equal to the Square of the Middle Term.

Thus in, \( a, ar, ar^2 \).

Tis Plain \( a \times ar^2 = ar \times ar = a^2 \times r^2 \).

**Problem I.**

To find any number (n) of Geometrical Means between any two given Quantities, \( a \) and \( e \).

Since, \( a, ar, ar^2, ar^3, \&c. ar^{n-1} = e \)

Therefore, \( r^{n-1} = \frac{e}{a} \), And \( r = \left( \frac{e}{a} \right)^{\frac{1}{n-1}} = \text{Ratio} \).

Or, putting \( x = \text{First Mean} = \text{Second Term} \).

Then, \( a, \frac{x^2}{a}, \frac{x^3}{a^2}, \frac{x^4}{a^3}, \&c. \frac{x^{n-1}}{a^{n-2}} = e \)

Therefore, \( x^{n-1} = a^n e \), And \( x = a^n e^{\frac{1}{n-1}} \).

Now, having found either the Ratio or First Mean, the rest are soon got.
SYNOPSIS

Part 1. Sect. 3.

PROBLEM II.

Given, any two Quantities, a and x.
Required, a 3d, 4th, 5th, or nth, Geometric Proportion.

'Tis evident from the Preceding Problem, that
\[ \frac{a^2}{x^2} \text{, or } x^{n-1} \] is the Proportional required.

COROLLARY 2.

If out of a Rank of Continued Geometric Proportionals, there be taken any Series of Equidistant Terms, that Series shall be also Proportional.

Thus,

If, \(a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, ar^7, ar^8\)
Then
\[ \left\{ \begin{array}{l}
a, \ar, \ar^2, \ar^3, \ar^4, \ar^5, \ar^6, \ar^7, \ar^8 \\
a, \ar, \ar^2, \ar^3, \ar^4, \ar^5, \ar^6, \ar^7 \\
a, \ar, \ar^2, \ar^3, \ar^4, \ar^5, \ar^6 \\
\end{array} \right\} \]

THEOREM 2.

If 4 Quantities be Geometrically Proportional; The Product of the Extremes, is equal to that of the Means.

That is, if \(a, ar, ar^2, ar^3\), Continued
\(a, ar, c, cr\), Discontinued

'Tis plain that
\[ ar \times ar^3 = ar \times ar^2 \quad (= a^2 r^3) \]
\[ ar \times cr = ar \times e \quad (= acr = are). \]

Or,
Or, if \( 3 : 5 :: 2 \times 3 : 2 \times 5 \)
\( \text{i.e. } 3 : 5 :: 6 : 10 \)

Then \( 3 \times 10 = 5 \times 6 \)
\( \text{i.e. } 3 \times 2 \times 5 = 5 \times 2 \times 3 \times \frac{2}{3} = 30 \)

**CONVERSE.**

If 2 Products arising from the Multiplication of two Quantities be equal;
Those 4 Quantities will be Proportional.

**CASE I.**

In any two Products, if the Factors of the one be made the 1st. Antecedent, and the 2d. Consequent; those of the other, the 1st. Consequent, and 2d. Antecedent, then the Terms are said to be Directly Proportional.

That is, if \( A \cdot b = a \cdot B \).

\[
\begin{align*}
A : B :: a : b, & \text{ or } A : a :: B : b. \\
B : A :: b : a, & \text{ or } B : b :: A : a. \\
a : b :: A : B, & \text{ or } a : a :: b : B. \\
b : a :: B : A, & \text{ or } b : B :: a : A.
\end{align*}
\]

**SCHOLIUM.**

If in 4 Quantities directly Proportional \((A : a :: B : b)\) any Three be given, the Fourth is readily found.

For \( A \cdot b = B \cdot a, \text{i.e. } 1\text{st.} \times 4\text{th.} = 2\text{nd.} \times 3\text{rd.} \) by this Theorem.

Therefore
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\[
\begin{align*}
1 \text{st} &= \frac{2^d \times 3^d}{4^d} = 2d \times \frac{3^d}{4^d} = 3d \times \frac{2^d}{4^d} \\
2 \text{nd} &= \frac{1^a \times 4^d}{3^d} = 1^a \times \frac{4^d}{3^d} = 4^d \times \frac{1^a}{3^d} \\
3 \text{rd} &= \frac{1^a \times 4^d}{2^d} = 1^a \times \frac{4^d}{2^d} = 4^d \times \frac{1^a}{2^d} \\
4 \text{th} &= \frac{2^d \times 3^d}{1^a} = 2d \times \frac{3^d}{1^a} = 3d \times \frac{2^d}{1^a}
\end{align*}
\]

NOTE.

That \( \frac{B^a}{A} \) is a 4th Proportional to \( A, B, \) and \( a, \) may be proved also thus;

For since \( A : B : a : 1 : \frac{B^a}{A} \) by Def. of Division.

And \( A \times \frac{B^a}{A} (= B \times a \times 1) = B \times a \) by this Theor.

Therefore, \( A : B : a : \frac{B^a}{A}, \) by its Converse, \( q. e. d. \)

CASE II.

But if the Factors of the one be made the 1st Antecedent, and 1st Consequent; those of the other, the 2d Antecedent, and 2d Consequent, then the Terms are said to be Reciprocally Proportional.

That is, if \( A : B = a : B. \)

If in 4 Quantities Reciprocally Proportional (A : b :: a : B) any Three be given, the 4th is easily found.

For A b = a B, i.e. 1st. × 2d. = 3d. × 4th. by the Def. of Recipr. Proportion.

\[
\begin{align*}
\text{1st.} &= \frac{3^d \times 4^\text{th}}{2^d} = 3d. \times \frac{4^\text{th}}{2^d} = 4\text{th.} \times \frac{3^d}{2^d} \\
\text{2d.} &= \frac{3^d \times 4^\text{th}}{1^a} = 3d. \times \frac{4^\text{th}}{1^a} = 4\text{th.} \times \frac{3^d}{1^a} \\
\text{3d.} &= \frac{1^a \times 2^d}{4^\text{th}} = 1\text{st.} \times \frac{2^d}{4^\text{th}} = 2\text{d.} \times \frac{1^a}{4^\text{th}} \\
\text{4th.} &= \frac{1^a \times 2^d}{3^a} = 1\text{st.} \times \frac{2^d}{3^a} = 2\text{d.} \times \frac{1^a}{3^a}
\end{align*}
\]

COROLLARY 2.

In Ranks of Similar Proportions, the Sums or Difference of the Corresponding Terms shall be Proportional.

That is, if \( \left\{ \frac{a}{x} : \frac{a r}{x r} : \frac{e}{x} : \frac{e r}{x r} \right\} \&c. \)

Then \( a \pm x : a r \pm x r : e \pm x : e r \pm x r. \)

COROLLARY 3.

In two Ranks of Proportionals, the Products or Quotients of the Corresponding Terms will be Proportional.

That is, if \( \left\{ \frac{a}{x} : \frac{a r}{x s} : \frac{e}{x} : \frac{e r}{x s} \right\} \)

\[ \text{Then} \]
Then \[
\begin{align*}
\frac{e}{x} : \frac{er}{xs} : : \frac{e}{r} : \frac{er}{rs}
\end{align*}
\]

**Corollary 4.**
The Powers of Proportionals, are also Proportional.

That is,

\[
\begin{array}{c|c}
\text{Continued} & \text{Discontinued} \\
\hline
a, ar, ar^2, ar^3, \text{ etc.} & a, ar, ar^2, \text{ etc.} \\
\hline
a^2, a^2r, a^2r^2, a^3r^3, \text{ etc.} & a^2, a^2r, a^2r^2, a^3r^3, \text{ etc.}
\end{array}
\]

\[
\begin{array}{c|c}
\text{Cor.} & \text{Cor.} \\
\hline
a^3, a^3r, a^3r^2, a^3r^3, \text{ etc.} & a^3, a^3r, a^3r^2, a^3r^3, \text{ etc.}
\end{array}
\]

**Corollary 5.**
Products having the same Common Factor are as the other Factors.

\[
\frac{an}{en} : \frac{en}{me} : : \frac{a}{e}
\]

That is,

\[
\frac{anm}{enm} : \frac{enm}{emn} : : \frac{a}{e}
\]

For \(anxe = enxa\) and \(anmx \cdot e = enmx \cdot a\), etc.

**Scholiwm 1.**

Hence, The Product of any two Quantities \(a\) and \(e\), is a mean Proportional between the Squares of those Quantities.

Thus \(a : a \cdot e : : a \cdot e : e\).

For P. Merci and P. Barrens.

**Schol.**
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SCHOLIUM 2.

Therefore, between $a^n$ and $a^m$, there are $n-1$ Mean Proportionals Composed of two Ranks of Powers.

The one \textit{In-}, creating, from \{ $a^{n-1}$ to $a^n$ \},

\textit{other Do-} creating, from \{ $a^{n-1}$ to $a^n$ \}.

SCHOLIUM 3.

If \textit{Unit} be put for the 1st. Term, the Root for the 2d. and its Consequent Powers for the 3d, 4th, \&c. Terms (in order), they will be a \textit{Series of Geometric Proportionals}, whose \textit{Common Ratio is the Root of Second Term};

And their \textit{Exponents will be a Series of Arithmetical Proportionals}, (expressing the \textit{Dimension of each Term, or its Distance from Unity}) either \textit{Affirmative or Negative}, according as the \textit{Root is More or Less than Unit}.

\text{This} \{ $a^4$, $a^3$, $a^2$, $a$, \&c. \}\{ \text{Expon.} \}

\text{But} \{ $a^4$, $a^3$, $a^2$, $a$, \&c. \}\{ \text{Powers} \}

\text{Therefore,} $a^{-n} = \frac{1}{a^n}$.

\text{Hence it is, that a Quantity drawn into it self any \textit{Number of Times}, that \textit{Number more one} is the \textit{Exponent} thereof.}

\text{Therefore, to \{ Double \}\{ \text{Square} \}\{ \text{Powers} \}} the \textit{Index of any Power, is to Square \&c.}

\text{Conseq. to take \{ Triple \}\{ \text{Cube} \}\{ \text{Powers} \}} of the \textit{Index of any Power, is to Cube \&c.} \textit{Extract the Root of that Power.}

\text{Therefore,} $\sqrt[n]{a^n}$ \textit{is more Naturally express’d by} $a^\frac{n}{n}$. 

K 2 Or,
Or, Because the Root is the 1st. Geometrical Mean between 1 and the Power;

But \( \sqrt[n]{a^n} \) is the 1st. Geometrical Mean between the Powers 1 and \( a^n \) by Probl. 1. Ch. 3.\n
And \( \frac{n}{r} \) is the 1st. Arithmetical Mean between their Exponents \( o \) and \( n \) by Probl. 1. Ch. 2.

Therefore \( \frac{n}{r} \) is the Exponent of the Power.

That is, \( \sqrt[n]{a^n} \left( = \frac{1}{\sqrt[n]{a^{n-r}}} \right) = a^{\frac{n}{r}} \).

Also, \( \frac{1}{\sqrt[n]{a^n}} \left( = \frac{1}{\sqrt[n]{a^{n-r}}} \right) = a^{\frac{-n}{r}} \).

And \( \sqrt[n]{a^n} = \frac{a^n}{\sqrt[n]{a^{n-r}}} = a^n \times \frac{1}{\sqrt[n]{a^{n-r}}} = a^{\frac{n}{r}} \).

For the same Reason.

**Corollary 6.**

If \( A \cdot a \cdot a \cdot B \cdot b \), then \( A \cdot a \cdot B \cdot b = a \cdot B \cdot a \cdot b \).

Therefore, \( A : a : : B : b \), for \( A \cdot b \cdot m = a \cdot B \cdot n \).

**Corollary 7.**

If \( a : a \cdot r : : e : e \cdot r \).

Then 1. \( na : a : : ne : e \).

2. \( na : a \cdot r : : e \cdot r : n \).

3. \( a : a : : n : e \).

**Corollary 8.**

If \( A \cdot a \cdot A \cdot r \cdot B \cdot b \).

i.e. \( A : A \cdot r : : B : B \cdot r \).

Then
Then 1. \( AA : a : a :: AB : a : b \).
2. \( Aa : Ab :: A : b : Bb \).
3. \( AA : a : B :: AB : B : Bb \).
4. \( AB : A : A :: BB : B : Bb \).

**Corollary 9.**

If \( a : ar :: e : er \),
\( e : er :: e : e \).
Then \( a = e \): \( ar = er :: e : er \).

**Corollary 10.**

If \( \{ A : B :: a : b \} \) and \( \{ C : B :: a : d \} \), then \( A : C :: d : b \).

For \( Ab (\equiv B a) = C d \), by this Theor.
Therefore \( A : C :: d : b \), by its Converse.

**Corollary 11.**

In \( \{ A, B, C \} \) if \( \{ A : B :: b : c \} \) and \( \{ B : C :: a : b \} \),
Then \( A : C :: a : c \).

For \( Ab (\equiv B b) = Ca \), by this Theorem.
Therefore \( A : C :: a : c \), by its Converse.

**Corollary 12.**

If \( a : ar :: e : er \),
Then \( \{ a : ar \} :: e : er :: a : e :: ar : e \).

**Corollary 13.**

In any Number of Geometric Proporionals;

As one Antecedent is to its Consequent;
So is the Sum of the Antecedents, to the Sum of the Consequents.

\( i.e. \)
Synopsis Part I. Sect. 3.

i.e. If \( a, ar, ar^2, ar^3, \ldots \), \( a, ar, c, er, \ldots \). Continued,

Then,

\[
\frac{a}{ar} = \frac{a + ar + ar^2}{2ar + ar^2 + ar^3} = \frac{a + exr}{2ar + er}
\]

**Corollary 14.**

If \( A : a :: B : b \). Then

1. **Alternately,** \( A : B :: a : b \).
2. **Inversely,** \( a : A :: b : B \).
3. **Compound,** \( A + a : a : B + b : b \).
4. **Conversely,** \( A : A + a :: B : B + b \).
5. **Dividedly,** \( A - a : a :: B - b : b \).
6. **Mixtly,** \( A + a : A - a :: B + b : B - b \).

All evidently **Proportional,** since the **Products** of their **Extremes** and **Means** are the **same** with those of the given **Proportion.**

**Scholiu M.**

**Proportions** **Compounded,** **Converted,** **Divided** and **Mixt,** are also **Proportionals** when **Alterned** and **Inverted.**

**Corollary 15.**

If \( a : ar :: c : er \), \( e : or :: u : ur \)

Then \( a : ar :: u : ur \).
SCHOLIUM.

In \(\{\frac{A}{a}, \frac{B}{b}, \frac{C}{c}\}\) if \(\{\frac{A}{a} : \frac{B}{b} :: \frac{c}{a}, \frac{B}{b} : \frac{C}{c} :: \frac{a}{c}\}\)

Then \(\frac{A}{a} : \frac{C}{c} :: \frac{a}{c} : \frac{c}{a}\).

For \(\frac{A}{a} : \frac{B}{b} :: \frac{c}{a}\) by Altern.
And \(\frac{B}{b} : \frac{C}{c} :: \frac{a}{c}\) by this Cor.
Therefore \(\frac{A}{a} : \frac{C}{c} :: \frac{a}{c}\) by Altern.

COROLLARY 26.

If \(\{\frac{A}{a} : \frac{B}{b} :: \frac{C}{c}, \frac{D}{d} : \frac{E}{e} :: \frac{r}{s}\}\)

Then \(\frac{e}{s} :: \frac{u}{u}\).

DEFINITION IV.

The product of the like terms of any Ratio is called a Compound Ratio or a Ratio compounded of those Ratio's.

Thus, the Ratio \(\frac{\frac{a}{m}}{\frac{e}{n}}\) is compounded of \(\frac{a}{m}\) and \(\frac{e}{n}\), or of \(\frac{a}{n}\) and \(\frac{e}{m}\).

SCHOLIUM.

Whence, Compound Ratio's are produced by the Multiplication of the Ratio's compounding them;

That is, \(\frac{\frac{a}{m}}{\frac{e}{n}} = \frac{a}{m} \times \frac{e}{n}\).

For
For \( m \times n = e \times a e \) by Schol. 4. Ch. 5. § 2.

And \( m \times n : a : n : m \times e : a e \) by Conv. Theor. 2.

Therefore, \( \frac{m \times n}{m \times n} : \frac{a \times n}{m \times n} \times \frac{m \times e}{m \times n} = \frac{a e}{m \times n} \) by equal Divis.

i.e. \( \frac{a}{m} : \frac{e}{n} = \frac{a e}{m \times n} \) by Theor. 2.

**Definition V.**

Similar Products are those whose Corresponding Factors are Proportional.

**Theorem 3.**

All Similar Products are as the Terms of the Ratio of the Corresponding Factors, raised into the Product's Dimension.

Suppose \( ABC, \&c. \) and \( ab \&c. \), the Products, whose Number of Dimension is \( n \).

The Similar Factors being as \( r \) to \( s \).

That is

\[
\begin{align*}
A : a : r : s \\
B : b : r : s \\
C : c : r : s \\
\&c.
\end{align*}
\]

Therefore, \( ABC, \&c. : ab \&c. \&c. : r^n : s^n \) by Cor. 3. Theor. 2.

**Corollary 1.**

The Ratio of any two Products is Composed of the Ratio's of the Factors.
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COROLLARy 2.

And all Similar Powers are in a Ratio Compounded of their Roots rais'd into the Index of these Powers.

COROLLARy 3.

Also Ratio's Compounded of equal ones are equal to one another.

Thus, if \( \frac{a}{m} = \frac{c}{n} \), and \( \frac{x}{y} = \frac{u}{z} \), then \( \frac{a}{m} \times \frac{x}{y} = \frac{c}{n} \times \frac{u}{z} \).

THEOREM 4.

In any given Quantities, \((a, b, c, d, e)\) the Ratio of the Extremes is Compounded of all the Intermediate ones.

That is, \( \frac{a}{c} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{d}{e} = \frac{abcd}{bcd} \).

COROLLARy.

Therefore, If \( \frac{A}{B} : \frac{B}{C} : \frac{C}{D} \), Then \( A : D :: a : d \).

And \( \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \).

That is, \( \frac{A}{D} = \frac{a}{d} \), by this Theor.

Conseq. \( A : D :: a : d \) by Def. of Proport.
Definition VI.

All Ratio's compounded of 2, 3, &c. Equal ones, are called the Duplicate, Triplicate, &c. of any one of those Ratio's.

Thus, if \( \frac{a}{m} = \frac{e}{n} \), then \( \frac{ae}{mn} = \frac{aa}{mm} = \frac{ee}{nn} \) is Duplicate of \( \frac{a}{m} \), or \( \frac{e}{n} \).

Corollary.

Therefore, in any number of Continued Geometric Proportionals,

\[ 1, a, a^2, a^3, a^4, a^5, a^6, &c. \]

The Ratio of the \( 3 \text{rd.} \) to the \( 1 \text{st.} \) is \( \text{Duplicate} \) of that of the \( 4 \text{th.} \) to the \( 2 \text{nd.} \).

And consequently,

The Ratio of the \( 3 \text{rd.} \) to the \( 1 \text{st.} \) is \( \text{Duplicate} \) of that of the \( 5 \text{th.} \) to the \( 2 \text{nd.} \).

Scholem.

In any Series of Geometric Proportionals;

\[ a, ar, ar^2, ar^3, ar^4, ar^5, ar^6, &c. \]

As \( a^n : ar^n :: a : a \times r^n \), that is, As
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As any \textit{power} of the 1st. is to the like of the 2d.
So is the 1st. to the \( n + 1 \) Term:
Or, So is the 1st. 2d. 3d. &c. to the \( n \)th. Term from it.

\textbf{Lemma 1.}

\textit{To Multiply a Ratio by any Quantity, is to Multiply the Antecedent of that Ratio by that Quantity.}

That is, \( m \times \frac{A}{B} = \frac{mA}{B} \).

For \( B : A :: m : \frac{mA}{B} \) by Sch. to Conv. Theor. 2.

And \( B : A :: 1 : \frac{A}{B} \) by Def. of Divis.

Theref. \( 1 : \frac{A}{B} :: m : \frac{mA}{B} \) by Cor. 16. Theor. 2.

But \( 1 : \frac{A}{B} :: m : \frac{A}{B} \times m \), by Def. of Mult.

Conseq. \( \frac{A}{B} \times m = \frac{mA}{B} \).

\textbf{Theorem 5.}

Ratio’s \textit{having the same} Consequents \textit{are Directly as their Antecedents.}

That is, \( \frac{A}{B} : \frac{a}{B} :: A : a \).

For \( a \times \frac{A}{B} = \frac{aA}{B} \), and \( A \times \frac{a}{B} = \frac{Aa}{B} \)

\( \left\{ \begin{array}{l}
\frac{aA}{B} = \frac{Aa}{B}, \\
\text{or} \ A \times \frac{A}{B} = A \times \frac{a}{B}
\end{array} \right\} \) by Lem. 1.

But \( \frac{A}{B} \).

Theref. \( \frac{A}{B} : \frac{a}{B} :: A : a \) by Conv. Theor. 2.
Lemma 2.

If a quantity multiplies another, and the product be divided by the same, the quotient will be the quantity multiplied.

That is, \( \frac{an}{n} = \frac{anr}{nr} \), &c.

For \( 1 : a : : n : a \) by Def. of Muls.

And \( n : an : : 1 : \frac{an}{n} \) by Def. of Divis.

Therefore \( 1 : a : : 1 : \frac{an}{n} \) by Cor. 15. Theor. 2.

Conseq. \( a = \frac{an}{n} \), &c. by Def. of Proport.

Corollary.

Therefore if the terms of a ratio be multiplied or divided by the same quantity, the ratio will be the same.

That is, \( \frac{x}{n} = \frac{xz}{nz} \), &c.

For putting \( an = x \), then \( \frac{an}{n} = \frac{x}{z} \), and \( \frac{anz}{nz} = \frac{xz}{nz} \).

But \( \frac{anz}{nz} (= a) = \frac{anz}{nz} \) by this Lemma.

Therefore \( \frac{x}{n} = \frac{xz}{nz} \), &c. by Substitution.
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THEOREM 6.

Ratios having the same Antecedents are Reciprocally as their Consequents.

That is \( \frac{A}{B} : \frac{A}{C} :: C : B \).

For \( \frac{B}{B} = 1 \), and \( \frac{A}{C} = \frac{BA}{BC} \) by preced. Cor.

But \( \frac{AC}{BC} : \frac{BA}{BC} :: AC : BA \), by Theor. 5.

That is, \( \frac{A}{B} : \frac{A}{C} :: C : B \).

THEOREM 7.

If \( A : B :: a : b \),

Then \( nA : B :: na : b \) and \( \begin{cases} \frac{A}{n} : B :: \frac{a}{n} : b \\ A : \frac{B}{n} :: a : \frac{b}{n} \end{cases} \).

1. For \( A : a :: B : b \), by Altern.

And \( nA : na :: A : a \), by Cor. to Lem. 2.

Theref. \( nA : B :: na : b \), by Altern.

2. Also \( \frac{A}{n} : \frac{a}{n} :: A : a \) by Theor. 5,

Theref. \( \frac{A}{n} : B :: \frac{a}{n} : b \), by Altern.
Synopsi

Part 1. Sect. 3.

THEOREM 8.

Quantities Proportional to their Differences are Continually Proportional.

For if $a : a - b :: b : b - c$

Then $a - b : b - c :: b : c$, by Divid.

That is $b : a - b :: c : b - c$

And $b : c :: a - b : b - c$ by Altern. of $\frac{\text{Last}}{\text{First}}$

But $a : b :: a - b : b - c$ by Cor. 16. Theor. 2.

Therefor. $a : b :: b : c$. by Cor. 16. Theor. 2.

THEOREM 9.

If Quantities are Continually Proportional, their Sums or Differences are also in Continual Proportion.

For if $a : b :: b : c :: c : d :: d : e$, &c.

Then $a \pm b : b \pm c :: c \pm d :: d \pm e$.

And $\frac{a \pm b}{b \pm c} :: \frac{b \pm c}{c \pm d}$ by Altern. &c.

But $a : b :: b : c :: c : d$, &c. by Sup.

Therefor. $a \pm b : b \pm c :: c \pm d$, &c.

THEOREM 10.

If $A : B :: a : b$; then $A : AB \frac{1}{2} :: a : ab\frac{1}{2}$.

For $AA : AB :: a : b^2$ by Cor. Lem. 2.

And $AA : AB :: aa : ab^2$ by Cor. Lem. 2.

Therefor. $A : AB\frac{1}{2} :: a : ab\frac{1}{2}$, by Evolution.

CHAP.
Chap. 4. Palmariorum Matheseos.

CHA P. IV.

Of Harmonic and Contra-Harmonic Proportions.

DE F I N I T I O N I.

When 3 Terms are so disposed, that the
1st. ∞ 2d. ∞ 3d. ∞ 1st. : 3d.
they are said to be Harmonically Pro-
portional.

Thus, 10, 15, 30, are Harmonic Proportional.
And \( b^3 - bn, b^3 - n^3, b^2 + bn \), make an Harm. Prop.
For \( bn - n^3 : bn + n^3 : : b^2 - bn : b^2 + bn. \)
Also, 12, 6, 4, are Harmon. Proport.
So \( b^3 + 3bn + 2n^3, b^2 + 2bn, b^3 + bn \), are Har. Prop.
For \( bn + 2n^3 : bn : : b^2 + 3bn + 2n^3 : b^2 + bn. \)

COROLLARY.

Whence, If the 2 first Terms of an Harmonic Proportion be given, the 3d. is readily found.

For if \( a, b, c \), be Harmonically Proportional.
Then, \( a - b : b - c : : a : c \), and \( ac - bc = ab - ac. \)
Theref. \( ab = 2a - b \times c, \) and \( bc = 2c - b \times a. \)
Conseq. \( c = \frac{ab}{2a - b} \), and \( a = \frac{bc}{2c - b} \).

DEFI-
DEFINITION II.

When 4 Terms are so disposed, that the
1st. ∝ 2d. ∝ 3d. ∝ 4th. ∝ 1st.: 4th.
they are also Harmonically Proportional.

As 10, 16, 24, 60; For 10 ∝ 16 : 24 ∝ 60 : 10 : 60.

COROLLARY.

If the 3 first Terms of such an Harmonic Proportion be given, the 4th is easily found.

For if \( a, b, c, d \) be Harmon. Proportional.
Then \( a - b : c - d :: a : d \); and \( a \cdot d - b \cdot d = a \cdot c - a \cdot d \).

Therefor, \( d = \frac{a \cdot c}{2a - b} \), and \( a = \frac{b \cdot d}{2d - c} \).

DEFINITION III.

If the Terms of an Harmonic Proportion be continued, then 'tis called an Harmonic Progression.

Thus, Supposing \( S \cdot b \) to be the 2d. Term
And that the 1st. exceeds the 2d.

The Progression will be

\[
\begin{align*}
&b + d, b, \quad b^2 + bd, b^2 + bd, \quad b^2 + bd, \quad b^2 + bd, \quad b^2 + bd, &\text{&c.}
\end{align*}
\]

COROL.
COROLLARY 3.

Whence, if out of a rank of Harmonic Proportionals there be taken any series of equidistant Terms, that Series will be Harmonically Proportional.

SCHOLIUM.

'Tis observed also, that Harmonic Proportion has several other properties common with those of Arithmetic and Geometric Proportion.

DEFINITION IV.

When three Terms are so disposed, that the Diff. of the 1st and 2d : Diff. of the 2d and 3d :: 3d : 1st.

they are said to be in a Contra-Harmonic Proportion.

Thus, 6, 5, 3, and 12, 10, 4, are Contra-Harmonics.

For 6 — 5 : 5 — 3 :: 3 : 6; and 12 — 10 : 10 — 4 :: 4 : 12

Or supposing b greater than n.

If the 2d. Term be greater than the 1st:

Then \( b^2 + n^2, b^2 + n^2, b^2 + bn, \) are Contra-Harmonics.

For \( bn - b^2 : n^2 - bn :: b^2 + bn : b^2 + n^2. \)

But if the 1st. Term exceeds the 2d:

Then \( b^2 + bn, b^2 + n^2, bn + n^2, \) are Contra-Harmonics.

For \( bn - n^2 : b^2 - bn :: bn + n^2 : b^2 + bn. \)
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CHAP. V.

Of Fractions or Parts of Integers.

DEFINITION I.

ANY Whole Thing or an Unit, may be considered as Divided into any Number of Equal Parts, which have their Name from the Number contain'd in that Unit.

As if an Unit be conceiv'd to be divided into four equal Parts, those Parts are call'd Fourths, and are thus written, \( \frac{1}{4} \).

And any Number of those Parts, suppose three are thus expres'd, Three Fourths, or Three divided by Four, and thus written \( \frac{3}{4} \), and called a FRACTION.

SCHOLIUM 1.

Hence, the Number \( \frac{3}{4} \) the Line is Numerator \( \frac{3}{4} \) Denominator expressing some of all those Parts into which the Unit is Divided.

SCHOLIUM 2.

If the Numerator be greater than the Denominator, the Fraction is greater than Unit, and is called an Improper Fraction.

COROL.
C O R O L L A R Y 1.

Whence it follows, That a Fraction is but a Quotient, signifying a Part or Parts of an Unit, express'd by a Numerator as a Dividend and a Denominator as a Divisor.

Thus, one Third part of any Thing is the Quotient of that Thing Divided by Three, which by Common Division is express'd thus, \( \frac{1}{3} \).

Also \( \frac{1}{4} \) signifies Three Fourths parts of an Unit, or One Fourth part of Three Units.

For \( \frac{1}{3} \) fourth of \( \frac{3}{1} \) Units \( \frac{3}{2} \) is Thrice one Fourth of one Unit.

C O R O L L A R Y 2.

As the Numerator is to the Denominator; So is the Fraction to Unit.

For the Dividend is to the Divisor, As the Quotient is to Unit. by Def. of Division.

C O R O L L A R Y 3.

Fractions having the same Denominators are one to another as their Numerators.

Thus, \( \frac{4}{3} : \frac{5}{3} :: 2 : 3 \).

For \( 2 : \frac{3}{5} :: 5 : 1 \) by preced. Cor.

And \( 3 : \frac{4}{5} :: 5 : 1 \) \( \frac{4}{5} \)

Thereof \( \frac{3}{5} : \frac{3}{2} :: 2 : 3 \), by Cor. 16. Th. 2. Ch. 3
SYNOPSIS. Part I. Sect. 3.

Corollary 4.

As any Fraction is to Units, so is Units to the Reverse Fraction.

Thus, \( \frac{1}{4} : 1 : 1 : \frac{1}{4} \).

For \( \frac{1}{4} : 1 : 3 : \frac{4}{3} \) by Cor. 2.

And \( 1 : \frac{3}{4} : \frac{3}{4} : 4 \).

Therefore, \( \frac{1}{4} : 1 : 1 : \frac{1}{4} \) by Cor. 16. Th. 2. Ch. 3.

Lemma 1.

To multiply the Numerator is to multiply the Fraction; thus, \( \frac{2 \times 3}{5} = \frac{2}{5} \times 3 \).

For \( 5 : 2 : 3 : \frac{2 \times 3}{5} \) by Case 1 of Conv. to Th. 2.

And \( 5 : 2 : 1 : \frac{1}{5} \), by Def. of Divis.

Therefore, \( 1 : \frac{3}{5} : 3 : \frac{9}{5} \) by Cor. 16. Th. 2. Ch. 3.

But \( 1 : \frac{3}{5} : 3 : \frac{3}{5} \times 3 \), by Def. of Multi.

Therefore, \( \frac{2 \times 3}{5} = \frac{3}{5} \times 3 \).

Lemma 2.

The terms of a Fraction being multiplied or divided by the same quantity alter not its value.

Thus \( \frac{2 \times 3}{2 \times 4} = \frac{3}{4} \).

For \( 2 \times 3 (= 6) : 2 \times 4 (= 8) : 3 : 4 \), by Cor. 5. Th. 2.

Therefore, \( \frac{3}{4} = \frac{3}{4} \), by Def. of Proportion.

Also...
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Also, \( \frac{10}{5} = \frac{2}{3} \)

For, \( \frac{10}{5} : \frac{15}{9} :: 10 : 15 \), by Cor. 3, Def. 1.

But, \( \frac{10}{5} : \frac{15}{9} :: 2 : 3 \).

Therefore, \( \frac{5}{3} = \frac{5}{3} \), by Def. of Proportion.

Definition II.

This is not only an Unit that may be divided into any Number of Equal Parts, but also any of those Parts may be divided infinitely into others called Compound Fractions, and those again subdivided infinitely.

These things being thoroughly understood, all Operations relating to Fractions admit of very few or no Difficulties.

Definition III.

The changing of Quantities out of one Form or Denomination into another, (either for the more ease in Working, or Estimating of their Value) is called Reduction.
SCHOLIUM.

Since it often happens in Reducing, Adding, Subtracting, &c. Fractions, that they swell into too great Numbers, which are not so manageable as smaller ones; therefore we shall in the next place shew the way of Reducing them into their Least Terms, either before or after such Operations, as there is occasion; which is done by the help of the following

PROBLEM.

Two Numbers being given, to find their Greatest Common Measure, (i.e.) the Greatest Number that can Divide both without Remainder.

RULE.

Divide the Greatest by the Least, and that Divisor by the Remainder continually, till nothing remain, and the last Divisor will be the greatest Common Measure.

EXAMPLE.

If the Numbers be 152, and 184:
Their Greatest Common Measure is 8.

\[
\begin{array}{r}
152) 384(2 \\
304 \\
80(152(1 \\
80 \\
72(80(1 \\
72 \\
8(72(9 \\
72 \\
0
\end{array}
\]

For
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For, Suppose \( a \) and \( b \) were the Quantities, whose Greatest Common Measure is required.

Now suppose \( a = 4b + c \).
And \( b = 2c + d \).
Also \( c = 3d \).

Then \( b = 7d \), and \( a = 31d \), therefore \( d \) is the Greatest Common Measure of \( a \) and \( b \).

**NOTE.**

If \( 1 \) be the Greatest Common Measure, the Numbers are said to be Prime to one another.

**PROPOSITION 1.**

To reduce a Fraction into its Least Terms.

**RULE.**

Divide the Numerator and Denominator by their Greatest Common Measure, their Quotients will be a Fraction equivalent to the former, and in the Least Terms.

**Numerical EXAMPLE.**

If the given Fraction be \( \frac{75}{135} \).

The Greatest Common Measure is 15.

And \( \frac{15}{15} \) 75 (5
15) 135 (9

For \( \frac{5}{9} = \frac{75}{135} \) by Lem. 2.

And 15 is the Greatest Number that will Divide 75 and 135 without Remainder, by the foregoing Prob.

Therefore, \( \frac{5}{9} \) is the least Term \( \frac{75}{135} \) can be brought to.

**Note,**
NOTE.

When the Greatest Common Measure is 1, the Fraction is already in the Smallest Terms.
For dividing by 1 does not diminish them.

COROLLARY 1.

A Fraction whose Terms are even, may be abbreviated by a Continual Division by 2.

Thus, \( \frac{256}{384} = \frac{2}{3} \)

For \( \frac{256}{384} \)

\[ \begin{array}{c|c|c|c|c|c|c}
2 & 128 & 64 & 32 & 16 & 8 & 4 \\
3 & 192 & 96 & 48 & 24 & 12 & 6 \\
\end{array} \]

\( \frac{2}{3} \)

COROLLARY 2.

So also may any Terms, that are found to be Divisible by any other Digit.

Thus, \( \frac{162}{1296} = \frac{2}{16} = \frac{1}{8} \)

For \( \frac{162}{1296} \)

\[ \begin{array}{c|c|c|c|c|c|c}
3 & 162 & 54 & 18 & 6 & 2 \\
3 & 1296 & 432 & 144 & 48 & 16 \\
\end{array} \]

COROLLARY 3.

When both Terms have Cyphers adjoining, cut off equal Cyphers from both.

Thus, \( \frac{1000}{3000} = \frac{1}{3} \) for \( \frac{1}{300} = \frac{1}{3} \)

And \( \frac{1000}{1460} = \frac{5}{7} \) for \( \frac{100}{146} = \frac{5}{7} \)

These Corollaries are evident from Lemma 2.
Literal EXAMPLES.

\[ \frac{xn}{xy} = \frac{n}{y}, \quad \frac{25}{5x + 15a} = \frac{5}{x + 3a}, \quad \frac{x^2y^2}{xy^3} = \frac{y}{x} \]

This is evident from Cor. to Lem. 2. Sect. 3.

Also

\[ \frac{nn - naa}{nn + 2a + aa} = \frac{nn - na}{n + a}; \quad \frac{x^2 - y^2}{x^2 + 2xy + y^2} = \frac{x - y}{x + y} \]

And

\[ \frac{x^6 a^2 y^2 n^2 + 4x^6 a^2 n^3 m}{y^2 m^3 + 4nm^3 r^4} = \frac{x^6 a^2 n^2}{m^2 r^4} \]

PROPOSITION 2.

To reduce an Integer into an Improper Fraction.

CASE I.

When there is no Denominator assigned.

RULE.

Let the given Integer be a Numerator, and Unit be Denominator.

Thus, \( 2 = \frac{2}{1}; \quad 15 = \frac{15}{1}; \quad 500 = \frac{500}{1} \).

And \( \frac{n}{1}; \quad a + c = \frac{a + c}{1} \).

For Dividing by Units does not diminish the Value.
C A S E II.

Where there is a Denominator assign'd.

R U L E.

Multiply the Integer by the assign'd Denominator, the Product shall be the Numerator.

Numerical E X A M P L E.

If \( \frac{14}{9} \) be the given \( \frac{\text{Integer}}{\text{Denominator}} \).

Then \( 14 \left( = \frac{9 \times 14}{9} \right) = \frac{13 \times 14}{9} \).

For \( 126 : 9 : : 14 : 1 \) by Def. of Divis.

And \( 126 : 9 : : \frac{13 \times 14}{9} : 1 \) by Cor. 2. Def. 1.

Therefore \( 14 = \frac{13 \times 14}{9} \).

Or \( 14 = \frac{13 \times 14}{9 \times 1} = \frac{9 \times 14}{9} \pm \frac{13 \times 14}{9} \) by Lem. 2.

Literal E X A M P L E S.

\[ x = \frac{n \times x}{n}; \quad x + y = \frac{x + y}{n} \]

For \( x \times n = n \times x; \quad x + y \times n = x \times n + y \times n, \) &c.

P R O P O S I T I O N 3.

To Reduce mixt Fractions into Improper ones.

R U L E.
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RULE.

Multiply the Integer by the Denominator of the Fraction, and add the Numerator to the Product, subdividing the same Denominator.

Numerical EXAMPLES.

36\frac{1}{4} \left(= \frac{36 \times 4 + 3}{4} = \frac{144 + 3}{4} = \frac{147}{4}\right)

For the Unit here is consider'd as divided into 4 Equal Parts.

Therefore the Integers must be Multiplied by 4, to produce 4ths.
To which 3 fourths being Added, the Sum will be also 4ths.

Literal EXAMPLES.

1. \( r + \frac{2n}{a} = \frac{2a + 2n}{a} \)

2. \( a + n - \frac{nx}{4r} = \frac{a + ar + arn - nx}{4r} \)

PROPOSITION 4.

To Reduce an Improper Fraction into an Integer, or Mixt Fraction.

RULE.

Divide the Numerator by the Denominator, and the Quotient will be the Integer, or mixt Fraction required.

Thus, \( \frac{13}{2} = 14 \); and \( \frac{24}{7} = 36\frac{2}{7} \).

For
For $\frac{3}{12} : 1 :: 126 : 9$, by Cor. 2. Def. 1.

And $14 : 1 :: 126 : 9$, by Def. of Divis;

Therefor, $\frac{13}{2} = 14$.

Also, $\frac{a + \frac{n}{d}}{a} = \frac{a}{d} + \frac{n}{a}$; and $\frac{a + \frac{a + n}{m + n}}{a}$

**Proposition 5.**

To reduce a Fracion into its Equivalent that shall have any assign'd Denominator.

**Rule.**

Multiply the Numerator of the Fracion by the assign'd Denominator, and Divide the Product by the Denominator of the Fracion; the Quotient shall be the Numerator required.

**Example.**

To reduce $\frac{1}{4}$ into a Fraction, whose Denominator is 12.

$$\frac{1}{4} = \frac{4 \times 3 \times 12}{12} = \frac{4 \times 36}{12} = \frac{9}{12} = \frac{3}{4}.$$

For $4 : 3 :: 12 : \frac{3 \times 12}{4} = \frac{18}{4} = 9$.

Therefor, $\frac{1}{4} = \frac{4 \times 3 \times 12}{12}$, by Def. of Prop.

**Coroll.**
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Corollary.

By this Proposition, Fractions are reduced into their Known Parts of Time, Measure, Weights, Coins, &c. As also into Decimals, Sexagesimals, &c. and the contrary.

Thus, \( \frac{1}{4} \) hr. \( = \left( \frac{3 \times 20}{4} \right) = 15 \text{ s.} \)

And \( \frac{1}{2} \) Deg. \( = \left( \frac{2 \times 60}{3} \right) = 40^\circ \)

Also \( \frac{7}{11} \) Deg. \( = 38^\circ, 10\text{ m}, 54\text{ s}^\prime, 32\text{ s}^\prime, &c. \)

For \( \frac{1}{11} \) Min. \( = \left( \frac{7 \times 60}{11} \right) = 38^\prime, 10\text{ s}^\prime, &c. \)

\( \frac{2}{11} \) Min. \( = \left( \frac{2 \times 60}{11} \right) = 10^\prime, 10\text{ s}^\prime. \)

\( \frac{3}{11} \) Sec. \( = \left( \frac{10 \times 60}{11} \right) = 54\text{ s}^\prime. \)

\( \frac{6}{11} \) Thirds \( = \left( \frac{6 \times 60}{11} \right) = 32\text{ s}^\prime. \)

Et.

Scholium.

Hence also, To reduce a Fraction into its Equivalent, that shall have any assign'd Numerator.

Example.

To reduce \( \frac{3}{4} \) into a Fraction, whose Numerator is 9.

Then \( \frac{3}{4} = \left( \frac{\frac{1}{4}}{3} \right) \times 9 = \frac{3}{15} \).

For \( 3:4::9:9 = 12. \)
PROPOSITION 6.

To reduce Fractions of Different Denominators, into their Equivalents, which shall have the same Denominator.

RULE.

Multiply all the Denominators continually, for a Common Denominator, and each Numerator continually by the other's Denominators, for new Numerators.

Numeral EXAMPLES.

1. \( \frac{1}{3} \) and \( \frac{1}{4} \) make \( \frac{8}{12} \) and \( \frac{9}{12} \).

For \( \frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} \) by Lem. 2.

And \( \frac{4}{3} = \frac{3 \times 3}{3 \times 3} = \frac{9}{12} \) by Lem. 2.

2. \( \frac{1}{7}, \frac{1}{4}, \frac{1}{5} \), make \( \frac{5}{35}, \frac{5}{20}, \frac{6}{35} \), or \( \frac{2}{14}, \frac{1}{14}, 3 \frac{1}{14} \).

\[
\left\{ \begin{array}{l}
\frac{1}{2} = \frac{1 \times 4 \times 7}{2 \times 4 \times 7} = \frac{28}{56} = \frac{14}{28} \\
\frac{3}{4} = \frac{3 \times 2 \times 7}{4 \times 2 \times 7} = \frac{42}{56} = \frac{21}{28} \\
\frac{9}{7} = \frac{3 \times 2 \times 4}{7 \times 2 \times 4} = \frac{48}{56} = \frac{24}{28}
\end{array} \right. 
\]

by Lem. 2.

Literal EXAMPLES.

1. \( \frac{a}{c}, \frac{c}{n} \) make \( \frac{an}{cn}, \frac{cc}{cn} \).

2. \( \frac{a}{x} \)
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2. \( \frac{a}{x}, \frac{c}{m}, \frac{\varphi n}{q} \) make \( \frac{a m \varphi}{x m q}, \frac{c m \varphi}{x m q}, \frac{\varphi n m}{x m q} \).

\[
\begin{align*}
\frac{a \times m \times q}{x \times m \times q} &= \frac{a m q}{x m q} \quad (== \frac{a}{x}) \\
\frac{c \times m \times q}{m \times m \times q} &= \frac{c m q}{m m q} \quad (== \frac{c}{m}) \\
\frac{\varphi n \times m \times m}{\varphi m \times m \times m} &= \frac{\varphi n m}{\varphi m m} \quad (== \frac{\varphi n}{\varphi m})
\end{align*}
\]

Therefore they are Equivalent Fractions, by Prop. 1.

*S C H O L I U M.*

Hence, to find Two Integers, that shall be one to the other as two given Fractions.

Suppose the Fractions were \( \frac{1}{2} \) and \( \frac{3}{4} \).

Then \( \frac{1}{2} : \frac{3}{4} :: \frac{3}{4} : \frac{1}{2} \) by this Prop.

But \( \frac{3}{4} : \frac{1}{2} :: 14 : 15 \) by Cor. 3. Def. 1.

Therefor. \( \frac{1}{2} : \frac{3}{4} :: 14 : 15 \) by Cor. 16. Th. 2. Ch. 3.

**PROPOSITION 7.**

**Addition and Subtraction of Common Fractions.**

*Note,* If the Numerators are not equal, they must be reduced to such as have equal ones; then by this

**RULE.**

*The Sum, or Difference of the Numerators set over the Common Denominator, shall be the Sum or Difference of the given Fractions.*

Numeral
Numerical Example.

Let the fractions be $\frac{3}{5}$ and $\frac{2}{3}$.

Then $\frac{3}{5} \pm \frac{2}{3} = \frac{9 + 2}{9} = \frac{11}{9}$ or $\frac{11}{9}$.

For $\frac{3}{5} : \frac{2}{3} :: 5 : 2$, by Cor. 3. Def. 1.

And $\frac{3}{5} \pm \frac{2}{3} :: 5 \pm 2 : 2$, by Compound or Divid.

But $\frac{5 \pm 2}{9} : \frac{3}{5} :: \frac{5 \pm 2}{9} : \frac{2}{3}$, by Cor. 16. Theor. 2.

Conseq. $\frac{3}{5} \pm \frac{2}{3} = \frac{5 + 2}{9}$.

Literal Examples in Addition:

1. $\frac{aa}{e} + \frac{ee}{c} = \frac{aa + ee}{c}$.

2. $\frac{aa - ax}{a} + \frac{a}{a} = \frac{2aa}{a}$.

3. $\frac{aa}{n} + \frac{ee}{m} + \frac{xx}{x} = \frac{a^2 + e^2 + x^2}{n, m, x}$.

4. $\frac{125}{x^3 - 25x} + \frac{x - 25}{x^3 + 10x + 25} = \frac{x^3 + 30x^2 + 250x + 625}{x^3 + 5x^2 - 25x - 125}$

Literal Examples in Subtraction:

1. $\frac{aa}{c} - \frac{ee}{c} = \frac{aa - ee}{c}$.

2. $\frac{n^4 + a^3 m}{n c m} - \frac{a^3}{n c} = \frac{n n m}{c m}$.

3. $2 \& n$.
Chap. 5. \textit{Palmariorum Matheseos.} 97

\[ \frac{2an + n^2}{a+n} - n = \frac{an}{a+n} \]
\[ \frac{a^3 + n^3}{cx - x^3} - \frac{nnnn}{aax - aax} = \frac{a^2 + a^3 n^3 - n^4 x}{a^2 cx - a^3 x^3} \]

The \textit{Reason} is the same with that in \textit{Numbers}.

\textbf{PROPOSITION 8.}

\textit{Multiplication of Common Fractions.}

\textbf{RULE.}

Multiply the \(\frac{\text{Numerator}}{\text{Denominator}}\) for a new \(\frac{\text{Numerator}}{\text{Denominator}}\).

\textit{Numeral EXAMPLE.}

\(\frac{3}{4} \times \frac{2}{3} = \frac{6}{12} = \frac{1}{2}\).

For \(\frac{4}{3} \times \frac{2}{3} = \frac{8}{9}\), by \textit{Lem. 2}. And \(\frac{5}{7} : \frac{2}{3} :: 6 : 8\) by \textit{Cor. 1. Def. 1}. That is, \(\frac{5}{7} : \frac{2}{3} :: 3 : 4\) Or, \(\frac{5}{7} : \frac{2}{3} :: \frac{1}{2} :: 1\)

Therefore, \(\frac{1}{4} \times \frac{1}{3} = \frac{6}{12}\), by the \textit{Nature of Proportion}

\textbf{NOTE.}

1. \textit{Mixt Fractions}, are to be \textit{Multiplied}, \textit{reduced} to \textit{Improper Single}.

2. If one of the \textit{Factors} be a \textit{whole Number}, it must be made an \textit{Improper Fraction}.

\textit{Literal EXAMPLES.}

\(\frac{a}{n} \times \frac{e}{m} = \frac{ae}{nm}\)

For
For if \( x = \frac{a}{n} \), and \( \zeta = \frac{c}{m} \), by Substit.

Then \( a = xn \), and \( \epsilon = \zeta m \), by Mult.

And \( \left( \frac{xn \zeta m}{nm} \right) = \frac{ae}{nm} (\therefore \zeta) = \frac{a}{n} \times \frac{c}{m} \)

2. \( \frac{a^2 + 2ab + b^2}{ab} \times \frac{bb}{a + b} = \frac{ab + b^2}{a} \)

3. \( \frac{an}{m} \times a = \frac{aan}{m} \); and \( \frac{aaa}{an + nn} \times a + n = \frac{a}{n} \)

4. \( a + \frac{cc}{a - c} \times a - c = a + c - a - c + cc \)

5. \( a + \frac{nn}{a - n} \times a - 2n + \frac{nn}{a} = a^2 - 2an + 2n^2 - \frac{n^3}{a} \)

**Scholium 1.**

The Product of any Quantity Multiplied by a Proper Fraction, is always Less than that Quantity.

For in Multiplying by an Unis, the Product will be equal to the Multipliand.

But a Less Multiplier gives a Less Product:

Therefore Multiplying by a Proper Fraction, (i.e. by Less than Unit.) the Product must be Less than the Multipliand.

**Scholium 2.**

Whence the Product of two Proper Fractions must be Less than either of them.

Thus,
Chap. 5. Palmariorum Matheseos. 99

Thus, \( \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \), and \( \frac{1}{3} \) is less than either \( \frac{1}{2} \) or \( \frac{1}{6} \).

For \( 1 : \frac{1}{2} :: \frac{1}{3} : \frac{1}{6} \), by Def. of Mult.

But \( \frac{1}{2} \times \frac{1}{3} \geq \frac{1}{6} \), therefor. \( \frac{1}{2} \times \frac{1}{3} > \frac{1}{6} \).

PROPOSITION 9.
Division of Common Fractions.

RULE.

Multiply the Dividend by the Divisor's Reverse Fraction; or, (which is the same) Imagine the Terms of the Divisor changed, then work as in Multiplication.

Numerical EXAMPLE.

Thus \( \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \),

That is, \( \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \).

For \( 1 : \frac{1}{2} :: \frac{1}{3} : \frac{1}{6} \), by Def. of Mult.

But \( \frac{1}{2} : 1 :: 1 : \frac{1}{6} \), by Cor. 4. Def. 1.

And \( \frac{1}{2} : 1 :: \frac{1}{3} : \frac{1}{6} \), by Cor. 19. Th. 2. Cb. 3. §. 3.

Therefor. \( \frac{1}{2} \div \frac{1}{3} = \frac{1}{6} \), by the Nature of Proporls.

NOTE.

1. If either Dividend or Divisor be whole, or mixt Numbers, or if both be Mixt Numbers; Reduce them to improper Fractions.

2. If they are Compound Fractions; Reduce them to single ones.

O 2 Literal
Synopsis Part 1. Sect. 3.

Literal Example.

1. \( \frac{a}{c} \div \frac{x}{n} = \frac{na}{nc} \); and \( \frac{ar}{m} \div \frac{m}{x} = \frac{nar}{mx} \).

2. \( \frac{n + 3}{5} \div \frac{2a}{3} = \frac{3n + 9}{100a} \); and \( \frac{aa}{n} \div \frac{aa}{n} = \frac{x}{n} \).

3. \( \frac{xxx}{nn} \div \frac{a}{an} = \frac{xxx}{an} \); and \( \frac{a}{nx} \div \frac{xxx}{nn} = \frac{an}{nn} \).

4. \( a - c \div \frac{a - c}{n} = \frac{a - c}{n} \); and \( \frac{nnxx}{r} \div \frac{nx}{r} = \frac{nx}{r} \).

5. \( \frac{a^2 + 2an + n^2 - ar - nr}{ar - nr - rr} \div \frac{a + n - r}{n + r} = \frac{a + n}{a - r} \).

Which may be demonstrated thus.

For \( \frac{x}{n} \div \frac{a}{c} = \frac{xc}{nc} \div \frac{an}{nc} \), by Prop. 6.

But \( \frac{xc}{nc} : \frac{an}{nc} :: x : an \), by Cor. 3. Def. 1.

i.e. \( \frac{x}{n} : \frac{a}{c} :: x : an \), by Equal Division.

Therefor. \( \frac{a}{c} \div \frac{x}{n} = \frac{an}{xc} \) by Def. of Propor.

Scholium 1.

If the Fractions are of one Denomination,

Then cast off that Denominator, and Divide the Numerators.

Because Fractions having the same Denominators, are as their Numerators, by Cor. 3. Def. 1.
Chap. 5. Palmariorum Matheseos.

SCHOLIUM 2.

The Quotients of any Quantity divided by a Proper Fraction is always Greater than that Quantity.

For in Dividing by Unit, the Quotient will be equal to the Dividend;

But a Less Divisor gives a Greater Quotient:

Therefore in Dividing by a Proper Fraction, (i.e. by Less than an Unit) the Quotient must be Greater than the Dividend.

Or thus, \( \frac{3}{4} ) \times \frac{6}{7} \) (\( = \frac{6}{7} \), And \( \frac{6}{7} \) is Greater than \( \frac{5}{7} \). For \( 1 : \frac{6}{7} \) :: \( \frac{1}{7} : \frac{5}{7} \), by Def. of Divis.

And \( 1 : \frac{5}{7} :: \frac{6}{7} : \frac{4}{7} \), by Alternation.

But \( 1 > \frac{5}{7} \), theref. \( \frac{6}{7} > \frac{5}{7} \).

PROPOSITION 10.

To reduce Compound Fractions into Single ones.

RULE.

Multiply \( \sum \) Numerators \( \times \) continually \( \sum \) Numerator.

the \( \sum \) Denominators \( \times \) for a new \( \sum \) Denominator.

EXAMPLE.

Thus, \( \frac{3}{4} \) of \( \frac{5}{7} \) being Reduced, is \( \frac{2 \times 4}{3 \times 7} = \frac{8}{3} \)

For if \( \frac{3}{7} = n \), then \( \frac{3}{4} \) of \( \frac{3}{7} \) is \( \frac{3}{4} \times n = \frac{3}{4} \times n \)

Therefore \( \frac{3}{4} \) of \( \frac{3}{7} \) = \( \frac{2 \times 4}{3 \times 7} = \frac{8}{21} \)

DEFINITION
Definition IV.

A Unit may be imagin’d to be equally divided into 10 Parts, and each of those into 10 more; so that by a continual Decimal Subdivision, the Unit may be supposed to be Divided into 10, 100, 1000, &c. equal Parts, call’d, 10th, 100th, 1000th, Parts of an Unit.

And since Integers increase from Unit, towards the Left-hand, in a Decuple Proportion, so that a Figure in any Place is Ten times as much as the same in the next Place below it, and but a Tenth part of what it signifies in the next Place above it; therefore as the 1st, 2d, 3d, &c. Place above that of Units, is Tens, Hundreds, Thousands, &c. So the 1st, 2d, 3d, &c. Place below that of Units, is Tenths, Hundredths, Thousandths, &c. Decreasing in a Subdecuple Proportion, as is evident from the following Table.

<table>
<thead>
<tr>
<th>Integers.</th>
<th>Parts.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bill. Million. Unit. Unith Millionth Bill.</td>
<td></td>
</tr>
<tr>
<td>x u c x u c x u c x u c x u c x u c x u c x u c x u c x</td>
<td></td>
</tr>
<tr>
<td>5 6 7 8 9 8 7 6 5 4 3 2 1, 2 3 4 5 6 7 8 9 8 7 6 5 8 9 8 7 6 5</td>
<td></td>
</tr>
</tbody>
</table>

Whence
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Whence those Fractions, whose Denominators are an Unit with a Cypher or Ciphers, are called Decimal Fractions, and may be written without their Denominators, distinguished by a Points or Comma prefix'd, and read like Integers, giving them the name of the last Place to the Right-hand.

Thus, \(\frac{5}{25}\) signifies \(\frac{5}{25}\) Tenths.

COROLLARY.

As \(\frac{5}{25}\) Ciphers before \(\frac{5}{25}\) Integers \(\frac{5}{25}\) Decimals, (advancing them so many Places towards the \(\frac{5}{25}\) Left \(\frac{5}{25}\) hand) \(\frac{5}{25}\) Increase \(\frac{5}{25}\) Diminish \(\frac{5}{25}\) their value.

\& C H O L I V M.

1. Hence it is, That 1, 2, 3, &c. Ciphers before a Decimal, advance it so many Places forward, whereby 'tis made 10, 100, 1000, &c. times less. Thus,

\[\frac{25}{\ldots}\] \(\frac{025}{\ldots}\) signifies \(\frac{25}{\ldots}\) Hundreds
\[\frac{0025}{\ldots}\] \(\frac{00025}{\ldots}\) signifies \(\frac{25}{\ldots}\) Thousandths

2. Therefore, a Figure in the 1st. 2d. 3d. &c. Decimal Place, is 10, 100, 1000, &c. times less than if it were an Integer.

3. Consequently, each removal of a Figure into a Place forward makes it Ten times less than it was before.
Proposition 11.

To reduce Vulgar Fractions into Decimals.

Rule.

To the Numerator add as many Ciphers as you would have Decimal Places; then divide it by the Denominator, and the Quotient (if there be no Remainder) will be a Decimal equivalent to the vulgar Fraction given.

But when there is a Remainder, you may by adding more Ciphers, proceed so as to bring the Quotient nearly equal.

Thus, \( \frac{1}{2} = \frac{1.0}{2} = .5 \), for \( 2:1::1.0:5 \)

\( \frac{1}{4} = \frac{3.00}{4} = .75 \), for \( 4:3::1.00:75 \)

\( \frac{2.000000\text{&c.}}{7} = .285714 = \frac{2}{7} \) near, not wanting 1\text{&c.} part of an Unit.

For 7:2::1.000000, \&c.: .285714, \&c.

SC H Q L I U SM.

'Tis observed when a Fraction is reduced to the smallest Terms:

That if its Denominator be Compounded only of the Prime Numbers 2 and 5 (the Components of 10) the Decimal of that Fraction will be determin'd.

But if the Denominator be Compounded of any other Prime Numbers, it will be Indetermined: and the same Figures will return again in order, and continue to Circulate, either by one Figure, or by two, three, \&c. Figures, tho' never by more than the Number of Units in the Denominator less 1.
Chap. 5. *Palmariorum Matheseos.*

For the *Remainder* being always less than the *Divisor*, therefore may be any *Number* less by 1 than it.

But in so many Operations, at most, as there are *Units* in the *Divisor*, one of the *Remainders* must return again;

Therefore, the same Figure in the *Quotient* must also return, and so continue the *Circulation*.

To find the *Number* of the *Circulating Figures*.

1. Divide the *Denominator* by 2 and 5 as often as possible; if it come to be 9, 99, 999, 9999, &c. or an *Aliquot part* of such *Number*, or a *Number Compounded* of 2 or 5, and such *Aliquot Part*; Then the *Number* of the *Circulating Figures*, will be equal to so many Figures of 9, as there are in the *Number* found.

2. If one of the *Prime Numbers* compounding the *Denominator* (excluding those of 2 and 5) be not an *Aliquot part* of the other:

Then the *Number* of the *Circulating Figures* will be equal to the *Product* of them required by those Compounding *Prime Numbers*.

3. And when the *Denominator* is *Compounded* of 2, or 5, or any Power of them; Then the *Circulating Figures* begin at such a *Place* of Decimals, as is denoted by the *Index* of 2 or 5, assumed in that *Composition*, more 1.

**Proposition 12.**

Addition and Subduction of Decimal Fractions.
RULE.

Place every Figure under that of the like Name, and Add, or Subduct, as if they were Integers.

Thus,

To 34.25 From 16.5
Add 3.026 Subd. .125

Sum 37.276 Rem. 16.375

The Reason is the same with that of Integers.

PROPOSITION 13.

Multiplication of Decimal Fractions.

RULE.

Multiply the Factors as if all were Integers; And the Decimals in the Product, must be equal to the Sum of those in both Factors; if they are not, prefix Cyphers to supply the Defect.

For the Index of each Figure in the Product, must be equal to the Sum of the Indices of the Multiplied and Multiplying Figures.

Thus,

Mult. 3.52 by 4.3, the Product is 15.136

For \( \frac{3.52}{4.3} = \frac{352}{43} \) and \( \frac{352}{43} \times \frac{43}{10} = \frac{15136}{1000} = 15.136 \)

Also, \( .013 \) Multi. by \( .005 \), gives \( .000065 \)

For \( \frac{.013}{.005} = \frac{13}{5} \) and \( \frac{13}{5} \times \frac{5}{1000} = \frac{65}{10000} = .000065 \)
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SCHOLIUM 1.

When a Decimal, or Mix'd Number is to be Multiplied by an Unit with Cyphers; 'Tis but removing the Point or Comma so many Places towards the Right-hand in the Multiplier, as there are Cyphers annex'd to the Unit.

Thus,

\[
\begin{align*}
10 & \quad 2.537 \\
100 & \quad 25.37 \\
1000 & \quad 253.7 \\
10000 & \quad 2537 \\
100000 & \quad 25370
\end{align*}
\]

SCHOLIUM 2.

In Multiplication, if it were required to find only an assigned part of the Product.

1. Set the Unit Place of the Lesser Number under that Place of the Greater, whose Index is equal to that of the assigned Right-hand Place of the Product (i.e. the first in Integers, and the last in Decimals) or to the Number of Figures to be cut off in the Integers, or left in the Decimals.

2. Then set the rest of the Figures of the Lesser Number in a contrary Order.

3. And begin to Multiply always at that Figure of the Multiplier, which stands over the Multiplier.

Note, That particular regard must be had of the Increase that would arise from the foregoing Figures of the Multiplier.

\[P_2\]
4. Set the Right-hand Places of the Products of each Figure in the Multiplier under one another, and their Sum will be the Product required.

**EXAMPLE.**

In Multi. 1098612286
By 4342945
Product: 4771214734412270

But suppose in this Product of 16 Places; the first 5 Places only were required, then by the foregoing Rule.

<table>
<thead>
<tr>
<th>Indices</th>
<th>1110</th>
<th>9876543210</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multipland</td>
<td>• •</td>
<td>1098612286</td>
</tr>
<tr>
<td>Multiplier</td>
<td>5 4</td>
<td>92434567892011</td>
</tr>
<tr>
<td>Indices.</td>
<td>0 1</td>
<td>234567891011</td>
</tr>
</tbody>
</table>

\[\begin{array}{c}
43944 \\
3296 \\
439 \\
22 \\
10 \\
\end{array}\]

\[47712 \text{ required.}\]
Chap. 5. *Palmariumum Matheseos. 109*

In the Prod. of $798.0625$

By $78.54$

Required the *Integers only*.

798.0625
4587

65864
6384
399
31

62679 Required.

In the Prod. of $419.3$

By $0.6375$

Reqd. the *Integers only*.

419.3
57360
251
13
3

267 Required.

In the Prod. of $798.0625$

By $78.54$

Reqd. *2 Decimal Places*.

798.0625
4587

6586437
638450
39903
3192

62679.82 Reqd.

In the Prod. of $535.625$

By $0.6375$

Reqd. *3 Decimal Places*.

535.625
57360
321338
16067
3749
267

341421

The *Reason* of this *Contradiction* is obvious:

For the *Index* of the *Right-hand Figure* of any *Product* is the *Sum* of the *Indices* of the *Factors*;

And by *inverting* the *position* of the *Figures* (as the *Rule directs*) the *Sums* of the *Indices* of each *Corresponding Place* in the *Factors*, will be equal among themselves; and therefore equal to the *Index* of the *Right-hand* *Place* of the *Product* required;

But
Synopsis

But Products, whose indices are equal, belong to the same place, therefore must be set under each other, and their sum must be the product required.

PROPOSITION 14.

Division of Decimal Fractions.

RULE.

Divide as if all were integers; (annexing cyphers to the dividend, if need be): And

Let the first figure in the quotient be of the same name (i.e. have the same index) with this figure of the dividend, which stands, (or is imagin'd to stand) over the unit place of the divisor.

For the index of each figure in the quotient, must be equal to the index of the divided figure less by the index of the dividing figure.

Or, the decimal places in the divisor and quotient must be equal to those in the dividend; if they are not prefix cyphers to the quotient, to supply the defect.

For the dividend is equal to the product of the divisor and quotient;

But both factors contain as many decimal places as the product does:

Therefore, what decimal places are in the dividend more than in the divisor, must be supplied in the quotient.

Thus,
Chap. 5. Palmariorum Matheseos.

Thus, .0325 divided by .25, gives .13 in the Quotient.

For \( \frac{.0325}{.25} = \frac{325}{25000} \) and \( \frac{25}{100} \cdot \frac{325}{10000} = \frac{32500}{1000000} = .13 \)

1. If the Divisor be Greater, or have more Decimal Places, than the Dividend; then by annexing Cyphers to the Dividend, the Quotient may be had to any accuracy. Thus, .25 \( \cdot .07864, 00 \cdot .31456. \)

2. Therefore, when there is a Remainder after Division, (tho' neither Dividend or Divisor consist of any Decimals) 'tis but adding Cyphers to the Dividend, and so proceed to any exactness.

3. When a Decimal or Mix'd Number is to be Divided by an Unit with Cyphers, 'Tis but removing the Point or Comma in the Dividend, so many Places further towards the Left-hand, as there are Cyphers annex'd to the Unit, prefixing Cyphers to the Dividend to supply vacancy, if need be. Thus,

\[
253.7 \text{ Divided by } \left\{ \begin{array}{c}
10 \\
100 \\
1000 \\
10000 \\
\end{array} \right\} \text{ Quotient is, } \left\{ \begin{array}{c}
25.37 \\
253.7 \\
2537 \\
0.2537 \\
\end{array} \right\}
\]

4. In Dividing by an Infinite Number, the Division may oftentimes be very usefully Contradicted, by Taking as many of the Left-hand Figures of the Divisor as you think convenient, for the first Divisor, by which Divide the given Number, and omit one Figure of the Divisor at each following Operation.

Thus,
PROP. 15.

To reduce a Decimal into a Common Fraction.

RULE.

Multiply the Numerator of the Decimal by the Denominator of the Common Fraction.

Thus, .575 of a l. is 11 s. 6 d.

\[
\begin{array}{c}
\text{.575 } \text{For } 1000 : \text{.575} : : 20s. : 11s. 15d. \\
\hline
20 \\
\hline
s. 11 \text{.500} \\
\hline
d. 6 \text{.000}
\end{array}
\]

And 10 : 5 :: 12 d. : 6 d.

PROP. 16.

To find a Multiplier that shall Effect the Same as a given Divisor.

CASE I.

If the Divisor be an Integer; make it a Denominator, and 1 the Numerator, that Fraction shall be the Multiplier required.

For
Chap. VI. The Arithmetic of Incommensurables.

DEFINITION I.

Those Quantities, which have the same Common Measure, are said to be Commensurable.

SCHOLIUM I.

Therefore Commensurable Quantities may be expressed by Numbers, or are as Number to Number.
DEFINITION II.

AND those Quantities which have not the same Common Measure, are said to be Incommensurable.

SCHOLIUM 1.

Therefore Incommensurable Quantities are not as Number to Number, that is, their Magnitudes cannot be express'd by Numbers, yet they may be Commensurable in Power.

SCHOLIUM 2.

Therefore the Roots of Imperfect Powers are Incommensurable Quantities, and are usually call'd Surd Roots, because inexpressible by any known way of Notation, otherwise than by their Radical Signs or Indices.

And tho' Incommensurable Quantities have no Ratio of Number to Number with Commensurable ones, yet they may be Commensurable one to another.

PROPOSITION 1.

To reduce Roots into their most Simple Expressions.

RULE.

Divide the given Power by the greatest Power, denoted by the Index, contained therein, that Measures it without Remainder; let the Quotients be affected by the Radical Sign, and have the Root of the Divisor prefix'd, or connected by the Sign X.

Thus, to reduce $a^{\frac{1}{2}}$ into its most Simple Expressions or Lowest Term.

Suppose
Chap. 6. *Palmariorum Matheseos.* 115

Suppose $x^n$ the greatest $n$ Power, that will Divide $a$ without Remainder.

Let $y = \frac{a}{x^n}$, then will $a^n = x \times y^{\frac{n}{n}}$.

For $a = x^y$, thereof $a^{\frac{n}{n}} = x^y$, $x^y = x^y$.

So $a^n = a^y - a^x y = a^x y$, $x^y - a^x y = a^x y$.

will be $\frac{a^n}{a + x - y x + y^{\frac{n}{n}}}$.

And, $a^n = 9 a^y + 27 a^x - 15 a^3 - 108 a^2 + 324 a - 324$.

will be $\frac{a^n}{a - 3 x^{\frac{n}{n}} + 12}$.

Also $75^{\frac{1}{2}}$ and $27^{\frac{1}{3}}$ will be $5 \times 3^{\frac{1}{2}}$ and $3 \times 3^{\frac{1}{3}}$.

For $\frac{75}{25} = \frac{27}{9} = 3$, by the Rule

Theref. $\frac{75^{\frac{1}{2}}}{5} = \frac{27^{\frac{1}{3}}}{3} = 3^{\frac{1}{2}}$, by *Evolution*.

Then, $75^{\frac{1}{2}} = 5 \times 3^{\frac{1}{2}}$, and $27^{\frac{1}{3}} = 3 \times 3^{\frac{1}{3}}$ by *Muh.*

**Proposition 2.**

To reduce Roots of Different Names, into those of the same Name.

**Rule.**

*Involve the Powers Reciprocally according to each others Index, for new Powers; And let the Product of the Indices, be the Common Index.*

Q.2 Thus,
Thus, \( a^{\frac{1}{n}} \) and \( e^{\frac{1}{m}} \) will be \( \sqrt[n]{a} \) and \( \sqrt[m]{e} \)

\( \sqrt[n]{a \times x} \) and \( \sqrt[m]{e \times y y y} \), will be \( \sqrt[n]{a} \times \sqrt[n]{x} \) and \( \sqrt[m]{e} \times \sqrt[m]{y y y} \)

Also \( b \times 4^{\frac{1}{n}} \) and \( c \times e^{\frac{1}{m}} \), will be \( b \times \sqrt[n]{4} \) and \( c \times \sqrt[m]{e} \).

This will be evident, when it is considered

That any \( \sqrt[n]{x} \), \( \sqrt[n]{x} = \frac{1}{2} \times \sqrt[n]{x^3} = \sqrt[n]{x} \), &c.

**COROLLARY 1.**

Hence, Rational Quantities may be reduced to the Form of any assign'd Root.

Thus, \( a \) reduced to the Form of \( x^{\frac{1}{n}} \) is \( \sqrt[n]{a} \).

**COROLLARY 2.**

Also, Roots with Rational Coefficients may hence be reduced so as to be wholly affected by the Radical Sign.

Thus, \( a \times e^{\frac{1}{n}} = \sqrt[n]{a} e \), and \( ay^2 \times x^{\frac{1}{n}} = x^\frac{1}{n} r a y y \).

**COROLLARY 3.**

Hence is known, whether any two given Roots are Commensurable one to another; As also, to find their Ratio.

For after Reduction into the lowest Terms, and the same Name;

If the Powers are equal, The Roots are Commensurable; And their Ratio is equal to that of the Rational Coefficients.
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Thus, \(\sqrt{75}\) and \(\sqrt{27}\) reduced, will be \(5\sqrt{3}\) and \(3\sqrt{3}\) which are Commensurable.

For \(5\sqrt{3} : 3\sqrt{3} :: 5 : 3\), by Cor. 5. Tb. 2. Ch. 3,

**PROPOSITION 3.**

Multiplication and Division of Simple Roots.

**RULE.**

Let the Product, or Quotient, of the Powers be affected by the Radical Sign, prefixing the Product, or Quotient, of the Rational Coefficients (if there be any).

**Examples in Multiplication.**

Thus, \(\frac{a^n}{e^n} = \frac{ae}{n^n}\); and \(\frac{a^n}{e^n} \times e^n = \frac{a e}{n^n}\)

Also, \(\frac{a^n}{e^n} \times e = \frac{ae}{n^n}\); and \(\frac{a^n}{e^n} \times e^n = \frac{a e}{n^n}\)

For \(1 : a :: e : a e\), by Def. of Multiplication.

But \(\frac{1}{n} \cdot \frac{1}{e^n} :: e^n : \frac{a e}{n^n}\), by Cor. 4. Theor. 2. Ch. 3.

Theref. \(\frac{a^n}{e^n} = \frac{a e}{n^n}\), by Conv. Theor. 2. Ch. 3.

**Examples in Division.**

Thus, \(\frac{a^n}{e^n} = \frac{a}{e}\), and \(b \frac{a^n}{e^n} = \frac{b}{e}\)

Also \(a \div e^n = \frac{a}{e^n}\), and \(\frac{a}{e} \div e^n = \frac{a}{e^n}\)

For
For \( e : i : ; a : \frac{d}{e} \) by Def. of Division.

But \( e^{\frac{i}{x}} : j^i : a^{\frac{d}{e}} : \frac{d}{e} \) by Cor. 4. Th. 2. Ch. 3.

Therefore, \( a^{\frac{i}{x}} \div e^i = \frac{a}{e} \) by Conv. Th. 2, Ch. 3.

**NOTE.**

If the Powers be the same; Then the Power affected by the Sum, or Difference of the Indices, shall be the Product, or Quotients sought.

In **MULTIPLICATION.**

1. \( y^n \times y^r = y^{n+r} \) and the \( \frac{n}{m} \) Power of \( y^{\frac{r}{s}} \) is \( y^{\frac{nr}{ms}} \)

2. \( \frac{y^i}{y^i} \times \frac{1}{y^i} \), or \( y^{-r} \times y^{-\frac{n}{m}} = y^{-\frac{m-n}{m}} = \frac{1}{y^{\frac{m-n}{m}}} \)

3. \( \frac{y^i}{y^i} \times \frac{1}{y^i} \), or \( y^{-\frac{n}{m}} \times y^{-\frac{r}{s}} \) will be

\[ y^{\frac{n+r}{m}} = \frac{1}{y^{\frac{n+r}{m}}} = \frac{1}{y^{\frac{n+r}{m}}} \] \( ; \) Also

4. \( \frac{a^i}{a^i} \times \frac{1}{a^i} \), or \( a^{-\frac{n}{m}} \times y^{-\frac{m-n}{m}} \times y^{-\frac{r}{s}} \) will be

\[ a^{-\frac{m-r}{s}} \times y^{-\frac{m+r}{s}} = \frac{1}{a^{\frac{m+r}{s}}} \] \( ; \) or

\[ \frac{1}{a^{\frac{m+r}{s}}} \times y^{\frac{m+r}{s}} \]
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In Division.

1. \( \frac{a}{y^m} \div y^r \) will give \( \frac{a}{y^{m-r}} \)
And \( y^{-\frac{1}{n}} y^m = y^{-\frac{1}{n}} \frac{m}{r} = y^{-\frac{m+n}{nr}} \)

2. \( \frac{1}{y^{\frac{1}{m}}} \) or \( y^{-\frac{m}{m}} y^{-\frac{n}{n}} = \frac{1}{y^{\frac{n}{m}}} \)

Proposition 4.

Addition and Subtraction of like Simple Roots.

General Rule.

Let the given Roots be Conjoined by the Sign +, or -.

1. But if the Roots are Commensurable; then,
Since in any two Quantities, \( a \) and \( e \), \( \frac{a}{e} + \frac{1}{e} \times e = a + e \).

Therefore,

\[
\begin{align*}
8^{\frac{1}{2}} + 2^{\frac{1}{2}} &= 18^{\frac{1}{2}} \\
18^{\frac{1}{2}} - 2^{\frac{1}{2}} &= 8^{\frac{1}{2}}
\end{align*}
\]

2. And if the Roots are reduced into their lowest Terms;
Then the Sum, or Difference, of the Rational Coefficients Multiplied by the Common Root, will be the Sum, or Difference, of the given Roots.

Thus, \( a \times r^{\frac{1}{n}} + b \times r^{\frac{1}{n}} = a \pm b \times r^{\frac{1}{n}} \)
For, if \( r^{\frac{1}{n}} = c \), then \( a \times r^{\frac{1}{n}} = a c \), and \( b \times r^{\frac{1}{n}} = b c \), theref.
\( a \pm b c = a \pm b \times c = a \pm b \times r^{\frac{1}{n}} = a \times r^{\frac{1}{n}} \pm b \times r^{\frac{1}{n}} \)

3. Also
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3. Also the Sum, or Difference, of any two Square Roots is equal to

The Square Root of the Sum, or Difference, between the Sum of the Powers, and twice the Product of their Roots.

Thus, \( \sqrt{75} + \sqrt{48} = \sqrt{243} \) or \( 9 \sqrt{3} \)
Also \( \sqrt{9} + \sqrt{4} = \sqrt{25} = 5 \)

For suppose \( \sqrt{a} \), and \( \sqrt{e} \) any given Quantities.

\[ \text{Tis plain, } a\sqrt{\frac{1}{e}} \pm e\sqrt{\frac{1}{a}} = a \pm e = a\sqrt{\frac{1}{a}} \pm \frac{2}{a} \sqrt{ae} + e\sqrt{\frac{1}{e}} \]

NOTE.

The Operations of Compound Surds are so easily deduced from what has been said of Simple ones, that we need not insist on them.

---

Chapter VII.

The Method of Resolving Mathematical Problems.

General Directions:

I. IT is absolutely necessary to have a clear and distinct Conception of the Conditions of the Question proposed to be Resolved.

II. Then substitute some Character or Letter, for each Quantity concerned in the Question.

And because in Algebraic Operations, the known and unknown Quantities are oftentimes very much Complicated; 'tis usual, for easier distinction, to put the first Letters of the Alphabet, for known, and the latter for unknown Quantities.
But if Quantities of different Species be Noted by the first Letter of their Names (when it can conveniently be done) the Expression will then be so Simple, so readily known and remembered, as to be of vast help and advantage to the Imagination; besides, it shortens the Process, yet preserves it free from all obscurity; and at the same time, wonderfully facilitates the Solution itself, as the Ingenious Learner will soon find by Practice.

III. By due reasoning from the Conditions of the Question, let the Quantities concerned there, in be justly Stated, and carefully compared; so that their relation to one another may appear, and the difference which renders them unequal, be discovered; and consequently the same Thing found expressible two ways or brought into an Equation, or into several Equations independent on each other.

NOTE.

1. If there are as many Equations given, as required.

Then the Question has a determinable Number of Solutions;

Because, each Quantity therein has but one value.

As suppose a Question proposed concerning the Age of 3 Persons, was thus Condition'd, viz.

The 2d. is 7 Years older than the 3d. The Age of the 3d. is Triple this of the 1st. and 2d. And the Sums of their Age is 68 Years; required the Age of each?

Put x for the Age of the 1st.

Then \(3x + 7, 6x + 21\) is the Age of the 1st, 2d. and 3d.

And \(x + 7 + 6x + 21\), or \(8x + 28 = 68\), by the conditions of the Question.
So that there is but one Equation given, and one required.

2. If there are more Equations given than required. Then the Equations are superfluous, and may sometimes happen to be contrary to, or inconsistent with the rest; and by that means, not only limit, but render the Question incapable of any Solution.

3. If there are more Equations required than given. Then the Question is Imperfectly determined; because capable of an Infinite Number of Answers. As suppose a Question proposed concerning the Age of three Persons; thus condition'd, viz.

The Age of the first is equal to the square of the Age of the second.

And the Sum of the Age of the 1st. and 2d. is equal to the Age of the 3d.

Required the Age of each.

Put $x, y, z$, for the Age of the 1st, 2d, and 3d Person.

Then $x = y^2$, and $x + y = z$, by the Conditions of the Question.

So that there are two Equations given; and three required:

Therefore the Question is Imperfectly determined;

For any one of these Quantities may be assumed at Pleasure.

IV. If the Question when stated, is found to have a determinable Number of Solutions.

Then the Equation directly drawn from the Conditions of the Question, (because at that time the known and unknown Quantities, are for the most part very much Compounded and mix'd together, on each side thereof) must be reduc'd into another, by Equal Augmentation and Diminution, where
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where only one Power of the unknown Quantity is found equal to some known ones; which is call'd a Simple or Pure, Compound or Affected Equation; And is then prepared for Solution, and distinguished by the Name of the Index of the Highest Power of the Quantity sought.

V. Let the Equation be cleared from all Fractions and Surd Roots, as also the highest unknown Power from any Coefficient.

VI. Let all the Terms on the Right-side be Transposed to the Left, so that the whole be made equal to nothing; And let the several like Terms be brought into one, by prefixing their Signs and Coefficients; placing the highest unknown Power first, the rest succeeding in Order, and the Absolute known Quantity last.

REDUCTION of EQUATIONS.

I. By Transposition.

That is, by Equal Addition, if the Quantity be Negative, and by Equal Subtraction, if Affirmative;

And is performed by transferring the Quantity to the other side of the Equation, with a contrary sign.

If \( x - 10 = 40 \)
Add \( +10 \)
\( +10 \)
Then \( x = 50 \)

i.e. \( x = 50 \)

If \( x + 10 = 40 \)
Sub. \( -10 \)
\( -10 \)
Or \( x = 30 \)

If \( z^3 - 3az^2 + b^2z = b^3 - b^2z + 2az^2 \)
\( -2az^2 + b^2z + b^2z = b^3 - 2az^2 \)

Then \( z^3 - 5az^2 + b^2z = b^3 \)

The Reason of this Operation is plain from Axiom 1.

R 2

II. By
II. By Equal Multiplication.

If \( \frac{x}{a} = b \), then \( \frac{x^3 + 3ax^2}{a^3} + \frac{a^2}{c^2} = a + a \).

Mult. by \( a \) · Mult. by \( c \)

Then \( x = ab \). Then \( x^3 + 3ax^2 + cn = c \cdot x \).

If \( \frac{x^3}{4} + ax^2 = \frac{c^2}{a} = 4a^2 \cdot c^2 \).

Then \( ax^3 + 4ax^2 - 4c^2 \cdot x^2 = 4a^2 \cdot c^2 \).

The Reason is evident from Axiom 2.

And by this means Equations are cleared from Fractions.

III. By Equal Division.

If \( ax = c \), then \( a + e \cdot a + e \).

Divid. by \( a \)

Then \( x = \frac{c}{a} \)

\( \frac{c}{a + e} \)

\( y = \frac{60 - 20}{5} = 8 \).

If \( x^4 + 2ax^3 = b^3 \), then \( x^2 + 2ax = b^3 \).

If \( x^4 - 2ax^2 = b^3 \), then \( x^2 - 2ax = b^3 \).

The Reason of this Operation is also evident from Axiom 2.

VI. By Equal Involution.

By means of which Equations are cleared from surd Quantities.

1. If \( ax^3 = b \), or \( ax^3 = c - b \).

Then \( ax = c^3 - 2cb + b^2 \cdot x \) by Squaring each side.
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2. If \( x - a = \frac{abx - a^2c}{x} \); Then,
\[ x^3 - 3ax^2 + 3a^2x - a^3 = abx - a^2c, \] by Cubing.

3. If \( x^3 - x\sqrt{a + \sqrt{b}} = 0 \), put \( \sqrt{a} = m \), and \( \sqrt{b} = n \).
Then \( x^3 = mx + n = c \). And \( m = \frac{x^3 + n}{x} \) by Tr. Inv. Di.

Therefor. \( m^2x^3 = x^6 + 2nx^3 + n^3 \), by Mult. and Invol.
And \( -n = \frac{x^6 - m^2x^3 + n^3}{2x^3} \) by Trans. and Divis.

Therefore, \( x^{1/2} = \pm \frac{m^2x^3 - 2n^2x^6 - m^4x^4 - m^2n^2x^3 + n^3}{x} \) by Mult. Invol. and Trans.

V. By Equal Evolution.

1. If \( z = 2z \), then \( z = \sqrt{2z} = 5 \).
2. If \( z^3 = 2z + a = c \), then \( x + a = c \).
3. If \( z^4 = 2za + a^2z = m^4 + n^3 \), Then,
\[ z^4 + a = m^2n, \] by Extr. the Sq. Root.

4. If \( z^6 = 3za^3 + 3a^2z^4 = m^4n^3 \),
Then \( z^4 - a = 2mn \), by Extr. the Cube Root.

5. If \( a^3 + 2a = c \), then by Extr. the Sq. R. of the unknown side, we have \( x + a \), with \(-a^3\) remaining;
Therefore, \( x + a \left( = x^3 + 2ax + a^2 \right) = c^2 + a \).
And if \( x^3 - 4ax = c^2 \), then the Sq. R. of the unknown side will be \( x - 2a \), with \(-4a^3\) remaining; therefore,
\[ x - 2a \left( = x^3 - 4ax + 4a^2 \right) = c^2 + 4a^2 \].

Therefore this Operation may be performed, by Adding the Square of half the Coefficients to each side of such Equation; and is called (not improperly) Completing the Square.

6. If
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6. If \( y^2 + 4ay^3 + 4a^2y^2 = c^2y^2 - 2ac^2y + \frac{1}{2}c^4 \) then the square root of the unknown side will be \( y^2 + 2ay - \frac{1}{2}c^2 \), with \(-\frac{1}{2}c^4 \) remaining; therefore,

\[
y^2 + 4ay^3 + 4a^2y^2 - c^2y^2 - 2ac^2y + \frac{1}{2}c^4 \rightarrow \frac{2ac^2y}{2a^2c^2 + \frac{1}{2}c^4}\]

will be equal to \( 2ac^2y + \frac{1}{2}c^4 \).

That is \( y^2 + 2ay - \frac{1}{2}c^2 = 2ac^2y + \frac{1}{2}c^4 \).

... S C H Q I L I U M ...

Whence the Ten first Propositions of the second Book of Euclid appear at Sight:

By putting \( s = a + c \), then \( a = s - c \), and \( c = s - a \).

Propositions.

1. \( ns = na + nc \) by Equal Multiplication.
2. \( ss = sa + se \) by Equal Multiplication.
3. \( s^2 = a^2 + ac \), and \( sc = c^2 + ac \) by Eq. Mult.
4. \( ss = a^2 + 2ac + c^2 \) by Equal Involution.
5. \( \frac{1}{2}ss = \frac{1}{2}a^2 + ac, \) for \( \frac{1}{2}s^2 = \frac{1}{2}d^2 = \frac{1}{2}a^2 \).
6. \( sd + e^2 = a^2 \), for \( sd = a^2 - e^2 \).
7. \( 2sa + e^2 = s^2 + a^2 \), for \( 2e^2 = s^2 - 2sa + a^2 \).
8. \( 2se + a^2 = s^2 + e^2 \), for \( 2a^2 = s^2 - 2se + e^2 \).
9. \( s^2 + e^2 = 2sa + a^2 \), for \( a^2 + e^2 \).
10. \( s^2 + a^2 = 2sa + 2e^2 \), by Preceding.

And by the like Comparisons most of those Propositions given by Vieta, Oughtred, and others, with innumerable more of this kind, of excellent use in the various Parts of the Mathematics, are deduced with great Facility.
The Derivation and Composition of Equations.

Since any quantity composed of parts may be reduc'd into parts, it must evidently follow, that the original components or roots of all equations, may be either affirmative, negative, mind'd, or imaginarily.

And because in all prepared equations, any one of its constitutive roots may be put for the unknown quantity;

Therefore, all prepared equations are really, or imaginarily constituted by the products of so many affirmative roots with negative signs, or negative roots with affirmative signs, connected to the unknown quantity, as is denoted by the index of its highest power; thus in

The Originations of Quadratic Equations:

1. If \( x^2 + ax + c = 0 \) then \( x^2 - ax + a, c = 0 \)

Therefore \( \overline{x^2 + ax + c = 0} \)

Put \( s = a + c \), then \( x^2 - sx + ac = 0 \)

2. If \( x^2 - ax + a = 0 \) then \( x^2 + sx + ac = 0 \)

Therefore \( \overline{x^2 + ax + c = 0} \)

Put \( s = a + c \), then \( x^2 + sx + ac = 0 \)

3. If
3. If \( \{ \begin{align*} x &= -a \\ x &= -e \end{align*} \) then \( \{ x = a \quad e = 0 \rangle \)

Therefore \( x^2 - ax - ae = 0 \) or \( +e \times \)

Put \( d = a \times e; \) if \( +a \) be \( \{ \text{greater} \} \) than \( -e. \)

Then \( x \times d - ae = 0 \)

Therefore, all Quadratic Equations are reducible to one of these Forms.

1. \( x^2 = dx + ae = 0 \)
2. \( x^2 = dx - ae = 0 \)

The Origin of any Cubic, Biquadratic, or higher Equation, may after the same manner be easily deriv'd.

From which Comparison 'tis evident that:

I. In every Prepared Equation Really constituted, which has, or is supposed to have, all its Terms;

The unknown Quantity has so many Roots or Values, as are the Dimensions of its highest Power; whereof so many are Affirmative as the Signs in Order have Changes, and the rest are Negative.

II. In such Prepared Equations, by changing the Signs in even Places;

1. The Affirmative Roots are made Negative, and the Negative Roots Affirmative.

Also, the Coefficients of the 2d. Term is the Sum of all the Roots so sign'd.

And the Coefficient of the 3d, 4th, 5th, &c. Term, is the Sum of the Products made of the Roots taken by 2's, 3's, 4's, &c.
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But the Number of Products, whose Sum is to be taken in the 3d, 4th, 5th, &c. Term is the unius of the 3d, 4th, 5th, &c. Term of a binomial rais'd to the Dimension of this Equation.

Therefore the Last Term or Absolute known Quantity is only the Product of all the Roots.

2. Hence, If all the Negative Products made of the Roots taken by 2's, 3's, 4's, &c. (Excluding their Signs), are equal to all the Affirmative ones, (the not respectively one to another) then the 3d, 4th, 5th, &c. Term is wanting.

And the contrary, If these are wanting, those must be equal.

PROBLEM 1.

To Increase, or Diminish the value of the unknown Roots of an Equation, by any given Quantity.

RULE.

Instead of the unknown Root, Substitute another Less, or Greater than it, by the Quantity given.

Exam. 1. To Augment the Roots of this Equation,

\[ xx + ax - bb = 0, \]

by the Quantity \( c \).

Put \( x = \alpha - c \), then you will have this Equation

\[ xx + a(x - c) - b(b - c) = 0, \]

whose Roots exceed those of the former, by the Quantity \( c \).

Exam. 2. To Diminish the Roots of this Equation,

\[ xx + ax - bb = 0, \]

by the Quantity \( c \).

Put \( x = \alpha + c \), then you will have this Equation

\[ xx + a(x + c) - b(b + c) = 0, \]

whose Roots are less than those of the Equation propos'd, by the Quantity \( c \).
COROLLARY 1.

To take away any one Term, except the First, from any given Equation.

Let the Equation be \( x^4 - ax^3 + bx^2 - cx + d = 0 \).

Put \( y + n = x \). Then

\[
\begin{align*}
y^4 + 4ny^3 + 6n^2y^2 + 4n^3y + n^4 &= x^4 \\
- a - 3an - 3an^2 - an^3 &= -ax \\
+ b + 2bn + bn^2 &= bx \\
- c - cn &= cx \\
+ d &= d
\end{align*}
\]

Now 'tis plain, That any Term, except the First, may be taken away from this Equation, because \( n \) was taken at Pleasure.

So that by putting \( 4n - a = 0 \), or \( n = \frac{1}{4} a \), the 2d Term must vanish;

And putting \( 6n^2 - 3an + b = 0 \), the 3d Term will also vanish:

After the same manner any other Term in this, or any other Equation may be destroy'd.

Therefore, in taking away the 2d Term of any Equation; Let the Index of the highest unknown Power be \( m \), and if the Coefficient of the 2d Term be \( \pm a \), substitute \( y = \frac{a}{m} \) in the room of \( x \) every where, then the 2d Term of that Equation will be destroy'd.

COROLLARY 2.

To add to an Equation any Term that is wanting.

Suppose the Equation was \( x^4 + cx^3 - d^4 = 0 \).

Put
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Put \( x = y - c \), then,

\[
y^4 - 4cy^3 + 6c^2y^2 - 4c^3y + c^4 = x^4 + x^3x + x^2 = x^3 + x
\]

\[
- d^4 = - d^4
\]

\[
y^4 - 4cy^3 + 6c^2y^2 - 3c^3y - d^4 = 0.
\]

PROBLEM 2.

To Multiply, or Divide the Roots of an Equation, by any given Quantity.

RULE.

Multiply, or Divide each Term respectively, by a Rank of Continual Proportionals from 1, whose Ratio is the Quantity given.

Exam. 1. To Multiply the Roots of this Equation,

\[
x^3 + ax^2 - bx^2 = ccd, \text{ by } 4;
\]

Mult. by \( \frac{1}{4} \frac{4}{16} \frac{16}{64} \) then,

We have \( x^3 + 4ax^2 - 16b^2x = 64c^3 \), an Equation whose Roots are Quadruple those of the former.

COROLLARY 1.

Hence, an Equation may be clear'd from Fractions; By Multiplying its Roots by the Product of the Denominators of the Fraction.

Thus, if \( x^3 - \frac{3}{4} ax^2 + \frac{3}{16} bbx - ccc = 0 \),

Mult. by \( \frac{1}{1} \frac{12}{144} \frac{1728}{1728} \)

Product; \( x^3 - 8ax^2 + 108bbx - 1728c^3 = 0 \).

NOTE.

If the Coefficient of the 2d Term of any Equation be not the Index of that Equation's Dimension, or some Multiple thereof, the Solution will be incumbered with \( \frac{S2}{9} \) Fractions;
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Fractions; to avoid which, Multiply all the Coefficients by a Rank of Continual Proportionals from 1, whose Ratio is the Equation's Dimension.

COROLLARY 2.

Also, Equations may by this Rule, sometimes be clear'd from Surd Quantities; By Multiplying the Roots of the Equation by the Surd Quantities to be clear'd.

Thus, \( x^4 + 2ax^3 \sqrt[3]{2} + 8bbx^2 - c^3x \sqrt[3]{8} = 2d^4 \)

\begin{align*}
\text{Multi. by } & \quad \sqrt[3]{2} & \quad 2 & \quad \sqrt[3]{8} & \quad 4 \\
\text{Prod. } & \quad x^4 + 4ax^3 + 16bbx^2 - 8c^3x = 8d^4
\end{align*}

Exam. 2. To Divide the Roots of this Equation

\( x^3 + 6ax^2 - 18bbx - 162c^3 = 0 \), by 3;

\begin{align*}
\text{Divide by } & \quad 3 & \quad 9 & \quad 27 \\
\text{We have } & \quad x^3 + 2ax^2 - 2b^2x - 6c^3 = 0, \text{ an Equation, whose Roots are } \text{thirds of those of the former.}
\end{align*}

COROLLARY.

Hence also Equations are sometimes clear'd from Surd Quantities, without being rais'd to Higher Powers; By Dividing their Roots by the Surd Quantity to be clear'd.

Thus, \( x^4 - ax^3 \sqrt[3]{2} + 2bbx^2 - cx \sqrt[3]{8} = 2d^4 \)

\begin{align*}
\text{Divide by } & \quad \sqrt[3]{2} & \quad 2 & \quad \sqrt[3]{8} & \quad 4 \\
\text{Quotient. } & \quad x^4 - ax^3 + bbx^2 - cx = \frac{1}{2}d^4
\end{align*}

And
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And if $x^3 - a \times x \sqrt[3]{2} + b \times x \sqrt[3]{32} = c \times d^2$

\[ \text{Divide by } 1 \]

\[ \sqrt[3]{2} \quad \sqrt[3]{4} \quad 2 \]

\[ \text{Quotient } x^3 - a \times x + 2b \times x = \frac{1}{2} c \times d^2. \]

**THE SOLUTION OF EQUATIONS.**

**I. Of Simple or Pure Equations.**

That the Reader might know the manner of applying the foregoing Rules, we thought it necessary in this Place, to infer some of the easiest Questions we could find.

And if the Method of Solution here used be well observed, the young Learner will find no difficulty in resolving those that are more intricate.

**Quest. 1.** One being ask'd how old he was, Answer'd, if $\frac{3}{5}$ of my Age be Multiplied by $\frac{4}{5}$ of the same, the Product will be my Age; I demand his Age. Suppose it $x$.

Then $\frac{x}{20} \times \frac{5x}{8} = \frac{5xx}{160} = x$, by the Question;

And $160x = 5xx$, by Multiplication:

Therefore, $160 = 5x$, and $x = \frac{160}{5} = 32$, by Division.

**Quest. 2.** One ask'd a Shepherd to sell him a 1000 Sheep, who Answered, that he could not then, for he wanted of his Demands just as many, half as many, and $72\frac{1}{2}$ Sheep more than he had; How many had he? Suppose it $x$.

Then
Then \( x + x + \frac{1}{2} x + 72 \frac{1}{2} = 1000 \), by the Question;
And \( 2x + \frac{1}{2} x = 1000 - 72,5 = 927,5 \), by Transposition.

Therefor, \( 5x = 1855 \), and \( x = \frac{1855}{5} = 371 \), Sought.

**Quest. 3.** A Privateer running at the rate of 10 Miles an Hour discovers a Ship 6 Leagues off, making away at the rate of 8 Miles an Hour; I demand in how many hours can the Privateer come in with the Ship. Suppose in \( x \) hours.

Then \( 8x + 18 = 10x \), by the Question;
And \( 10x - 8x = 18 \), by Transposition;

Therefor, \( x = \frac{18}{10-8} = \frac{18}{2} = 9 \) hours Sought.

**Quest. 4.** The Age of two Persons A and B, being 100 years; the Age of A exceeds that of B, by 40 years; I demand the Age of each. Suppose them \( x \) and \( y \).

Then, \( \begin{align*} \frac{x + y = 100}{\frac{x - y = 40}{2}} \end{align*} \)
\( \therefore x = 100 - y \); \( \therefore y = 30 \), and \( x = 70 \).

Or thus,

Suppose \( x = \text{Age of } A \), then \( x - 40 = \text{Age of } B \),
And \( 2x - 40 = 100 \), by the Question, therefor. \( 2x = 140 \),

Conseq. \( x = \frac{140}{2} = 70 \), and the Age of \( B \) is 30.
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Ques. 5. The Persons A, B, C, owe me Money, but I forgot both the Sum and Particulars, yet by comparing some Accounts I have by me, it appears that the Debt of A and B is 47l. of A and C, 71l. of B and C, 88l.; I demand each Man's Debt. Suppose them x, y, and z.

Then
\[ \begin{align*}
x + y &= 47 ; \\
x + z &= 71 ; \\
y + z &= 88 ;
\end{align*} \]

Therefore:
\[ \begin{align*}
x &= 56l. \\
y &= 32l. \\
z &= 15l.
\end{align*} \]

Ques. 6. A Gentleman bought a House, a Park, and a Garden; their prices were as 12, 5, and 1; but the Sum of double the price of the House, triple the price of the Park, and quadruple the price of the Garden, is so much greater than 10000 Pounds, as the Sum of the price of the House and Park is less than 5000 Pounds; I demand what each cost.

The House cost 12x, Park 5x, Garden 1x²
Then 24x + 15x + 4x = 43x
And 43x - 10000 = 5000 - 17x
Therefore: 60x = 15000, by Transposition,
And \[ \frac{15000}{60} = 250 \], by Division,
Therefore the House cost 3000l., Park 1250l, Garden 250l.

NOTE.

That some of the preceding Questions with innumerable more such may be resolv'd much easier, by other Methods Peculiar to each;

But our design is only to acquaint the Reader with General ways of resolv'n such Questions, whence he may at Pleasure draw variety of Particular ones for his Practice.
And the Method of expressing each unknown quantity by an unknown letter, we take to be the most general; because it serves universally in all cases, and does not in the least appear as if conceived: and therefore is preferable to any other.

Besides, this way cannot possibly strain the imagination, nor mislead the fancy, in finding other particular ways of notation: For all the reasoning, it supposes, is no more than what is immediately founded on the conditions of the question; and all the deductions are naturally drawn, only by reducing the equation thus form’d, and substituting in the room of certain unknown quantities, their equivalents in different expressions.

Those that are able to judge what method of proceeding is the most natural will see that we introduce only that which ought to be known, and deserves attention; the which we have insisted on the more that the beginner might have clear and distinct ideas of, and be thoroughly acquainted with the principal foundation of all analysis, that he might hereafter the easier know how to draw useful inferences, and make methodical applications in the more abstract parts.

For analysis of all other methods is the most conducive to, if not the only means for the discovery of unknown truths; and not only the mathematics, but in general all other sciences would have been imperfect without it; and the more they observe its laws, the nearer they arrive to perfection.
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Of Indetermined Questions.

Quest. 7. A Composition is to be made of three ingredients; the whole being 10 Pound Weight, the particulars are worth 2s. 4s. and 3s. the Pound; 'tis required to mix them so, as that the Pound may be afforded for 5s. How much of each must be taken? Suppose x, y, z.

Then \[ x + y + z = 10 \] by the Conditions of the Quest.
And \[ 2x + 4y + 8z = 50 \] by the Conditions of the Quest.

Here 'tis plain, the Question is indetermined; because there are three unknown Quantities, and but two Equations.

Therefore let the Limits of any one of the unknown Quantities, suppose z, be determined, which may be done after this manner.

Since \[ x = 10 - y - z = \frac{50 - 4y - 8z}{2} \] by Tripos.
And \[ y = 15 - 3z \]; Therefore \( z < 5 \).

Also \( x = 10 - 15 - 3z = 2z - 5 \); \( z \geq 2 \frac{1}{3} \).

Whence these Answers in whole Numbers.

<table>
<thead>
<tr>
<th>z</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
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<tr>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Quest. 8. A Refiner has 3 Ingots of Silver of different fineness, viz. of 4, 6, and of 10 Ounces fine, of which he would mix 20 pound weight, so as to make it 8 Ounces fine; how much must be take of each sort? Suppose x, y, and z.

Then
Then \( x + y + z = 20 \) \( \frac{3}{5} \) by the Question;
And \( 4x + 6y + 10z = 160 \) \( \frac{5}{4} \).
Therefore \( x = \frac{160 - 6y - 10z}{4} \).
And \( y = 40 - 3z \), therefore \( z < 14 \).
Also \( x = 20 - 40 - 3z - z = 2z - 20 \) \( \therefore \) \( z > 10 \).
Whence these Answers in whole Numbers.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( y )</th>
<th>( x )</th>
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</thead>
<tbody>
<tr>
<td>11</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>6</td>
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</table>

Quest. 9. A Grocer having 4 sorts of Sugar, at 12 d. 8 d. 6 d. and 4 d. a Pound, desires to have a mixture of a 100 weight made out of them, so that it may be afforded at 10 d. a Pound: how many Pounds must be taken of each? Suppose \( x, y, z, u \).

Then \( x + y + z + u = 100 \) \( \frac{2}{3} \) by the Question.
And \( 12x + 8y + 6z + 4u = 10x + 100 \) \( \frac{5}{3} \).
Therefore \( y + z + u = 100 - x \) \( \frac{2}{3} \) by Transf.
And \( 8y + 6z + 4u = 1000 - 12x \) \( \frac{3}{5} \).
Then \( 4y + 4z + 4u = 400 - 4 \cdot \frac{2}{3} \) from \( x \) by \( \frac{3}{5} \).
And \( 8y + 8z + 8u = 800 - 8 \cdot \frac{2}{3} \).

Then \( \frac{400 - 4 \cdot \frac{2}{3}}{800 - 8 \cdot \frac{2}{3}} < 1000 - 12x \). \( \therefore \) \( x < 75 \).

Therefore supposing \( x \) (for instance) \( = 60 \); let it be substituted in the Equations which express the Conditions of the Question.

Then
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Then $60 + y + z + u = 100$
And $720 + 3y + 6z + 4u = 1000$

Therefor. $y = 40 - z - u = \frac{280 - 6z - 4u}{8}$

And $z' = 20 - 2u$, therefore $u < 10$

But $y = 40 - 20 - 2u - u = 20 + u$,
Therefore $u$ has no other Limits, but may be any Number under 10.

Hence, in taking $x = 60$, we have these Answers in whole Numbers.

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<tr>
<th>$x$</th>
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<td>27</td>
<td>28</td>
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<tr>
<td>$z$</td>
<td>18</td>
<td>16</td>
<td>14</td>
<td>12</td>
<td>10</td>
<td>8</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>$u$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
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And by taking $x$ = to some other Number between its respective Limits, more Answers in whole Numbers may be found.

SCHOOL.

From this and the foregoing Chapters the several Practical Rules in Common Arithmetic are naturally deduced.

I. The Single Rule of Three, Direct and Inverse.

Wherein there are three Terms given, two of which are of the same Denomination.

And in Questions falling under this Rule, the Terms may be disposed in this order, viz.

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<th>2</th>
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The Querying Term.
That of the same kind. in the \(\frac{3d.}{2d.}\) Place : Then.
And that of different kind.

If the \(\frac{\text{greater}}{\text{les}}\) than the 1st. and requires \(\frac{\text{greater}}{\text{les}}\) than the 2d.

That is, if \(\frac{\text{more}}{\text{less}}\) requires, \(\frac{\text{more}}{\text{less}}\)

The Proportion is Direct; But

If the \(\frac{\text{greater}}{\text{les}}\) than the 1st. and requires \(\frac{\text{less}}{\text{les}}\) that the 4th. should be \(\frac{\text{greater}}{\text{les}}\) than the 2d.

That is, if \(\frac{\text{more}}{\text{less}}\) requires \(\frac{\text{less}}{\text{more}}\)

The Proportion is Inverse; And

The Required Term is \(\frac{\frac{2d. \times 3d.}{1st.}}\) in the Direct Rule;
\(\frac{\frac{1st. \times 2d.}{3d.}}\) in the Inverse Rule.

But in the Inverse Rule, if the Terms be placed thus,

The Querying Term.
That of the same kind. in the \(\frac{1st.}{3d.}\) Place ;
That of different kind.

Then the Work is as in the Direct Rule.

**Corollary.**

When the Rule of Three Direct has 1 for the 1st. Term, 1s. usually called the Rule of Proportion, from its frequent use and ready Performances in Common Affairs; for,

1. If
1. If the Price, &c. of the Integer be an Alleque part of a Pound, Shilling, &c. as.

If 1 yd. = 6 d. :: 420 yds. = w ? But 6 4 = 6 \times \frac{1}{12} = \frac{1}{2}.

\[ w = \frac{\frac{1}{2} \times 420}{1} = \frac{420}{2} = 210. \]

If 1 yd. = 4 d. :: 63 yds. = w ? But 4 d. = \frac{4}{12} = \frac{1}{3}.

\[ w = \frac{\frac{1}{3} \times 63}{1} = \frac{63}{3} = 21. \]

If 1 yd. = 10 s. :: 34 yds. = w ? but 10 s. = \frac{10}{20} = \frac{1}{2}.

\[ w = \frac{\frac{1}{2} \times 34}{1} = \frac{34}{2} = 17. \]

If 1 yd. = 2 s. :: 56 yds. = w ? but 2 s. = \frac{2}{20} = \frac{1}{10}.

\[ w = \frac{\frac{1}{10} \times 56}{1} = \frac{56}{10} = 5.6. \]

2. If the Price, &c. of the Integer is Composed of the Alleque Parts of a Pound, Shilling, &c. as.

If 1 yd. = 9 d. :: 84 yds. = w ? but 9 d. (= \frac{9}{12} + \frac{3}{12}) = 2 s. + \frac{1}{4}.

\[ w = \frac{\frac{9}{4} \times 84}{4} = 42 + \frac{21}{4} = 42.5. \]

If 1 yd. = 7 s. :: 264 yds. = w ? but 7 s. (= \frac{7}{20} + \frac{2}{20}) = \frac{9}{4} + \frac{1}{10}.

\[ w = \frac{\frac{264}{4} + \frac{264}{10}}{1} = 66 + 26 + \frac{4}{5} = 92\frac{4}{5}. \]

3. Also
3. Also if 1 lb. = \( \pi \) f. :: 1 C. (= 112 lb.) \( \pi \)?
Since 112 f. (= 2 \( \pi \) + 4 d) = 2 \( \pi \) + 1 gros,
Therefore \( \pi = n \times 2 \pi + \pi \) sought.

The manner of Operation in the other Cases of the Rule of Practice cannot but appear evident to them that throughly understand what is here already delivered, therefore this alone was thought sufficient; especially, since that Accurate Penman and most Ingenious Accompanist Mr. George Shelley, in his late Supplement to Mr. Wingate's Arithmetick, has (amongst variety of very useful Compendious Rules for the ready Solving of such Questions, as daily occur in Common and Ordinary Affairs) particularly insinced on the several parts of the Rule of Practice, the which he that would be farther inform'd would do well to consult.

II. The Compound Rule of Three, Direct and Inverse.

Let the Terms of the Supposition with their corresponding ones of the Demand, be so placed, as to appear directly Stated; then 'twill be either:

\[
\begin{align*}
1 & \quad 2 & \quad 3 \\
& a, & b, & c, & c, & b, & c, & & \\
\text{where } c &= \frac{c}{A} \frac{A}{B} ; & b &= \frac{b}{C} \frac{C}{a} ; & a &= \frac{a}{C} \frac{c}{A} \frac{A}{B} ;
\end{align*}
\]

For
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For

1. \( \frac{A: C \cdot a : w}{B : w \cdot b : c} = \frac{AB \cdot a \cdot b}{C} \)
2. \( \frac{C: A \cdot c : w}{a : w : B : b} = \frac{cAB}{Cz} \)
3. \( \frac{C: A \cdot c : w}{b : w : B : d} = \frac{cAB}{Gz} \)

**Example 1.**

If the \( m \) Men \( m \) in the Time \( t \) spend Pounds \( P \)?

Then \( MT : P : m : t : p = \frac{Pmt}{MT} \)

For \( M : m : P : \frac{Pm}{M} \)

And \( T : t : \frac{Pmt}{M} : \frac{MT}{P} = p \) sought.

**Example 2.**

If a \( l \) Body \( l \) Long, \( b \) Broad, \( T \) Thick, \( W \) Weighs \( W \)?

Then \( LBT : W : lb : \frac{Wlb}{LBT} = W \)

For \( L : l : W : \frac{Wl}{L} \)

Also \( B : b : \frac{Wl}{L} : \frac{Wlb}{LB} \)

And \( T : t : \frac{Wlb}{LB} : \frac{Wlb}{LBT} = W \) sought.

III. The
III. The Rule of Fellowship.

Example in the Single Rule of Fellowship.

Suppose \( \sum_{M} \frac{M}{m} \) put \( \sum_{L} \frac{L}{l} \) into the Joint Block;

Let \( x \) be the whole Gain; or Loss:

Required the Gain, or Loss of each of them. Then,

\[
\begin{align*}
L : \frac{L}{L+L+\frac{l}{l}, \text{ &c.}} &= M \text{'s} \\
L+L+\frac{l}{l}, \text{ &c.: } x \ldots &= \frac{Lx}{L+L+\frac{l}{l}, \text{ &c.}} = M \text{'s} \\
\text{&c.} \\
\end{align*}
\]

Gain, or Loss.

For \( L+L+\frac{l}{l}, \text{ &c.: } x \ldots L+L+\frac{l}{l}, \text{ &c.: } \frac{Lx+Lx+Lx}{L+L+\frac{l}{l}, \text{ &c.}} = x \).

Example in the Compound Rule of Fellowship.

Sup. \( \sum_{M} \frac{M}{m} \) put \( \sum_{L} \frac{L}{l} \) for the Time \( \sum_{T} \frac{T}{t} \); Then,

\[
\begin{align*}
LT : \frac{LT}{LT+LT+\frac{lt}{t}, \text{ &c.}} &= M \text{'s} \\
LT+LT+\frac{lt}{t}, \text{ &c.: } x \ldots &= \frac{LTx}{LT+LT+\frac{lt}{t}, \text{ &c.}} = M \text{'s} \\
\text{&c.} \\
\end{align*}
\]

Gain, or Loss.

For \( LT+LT, \text{ &c.: } x \ldots LT+LT, \text{ &c.: } \frac{LTx+LTx}{LT+LT, \text{ &c.}} \).

IV. of
IV. Of Reduction and Exchanges.

Ex. 1. If 100 Ells of Antwerp = 75 Yards of London; How many Yards of London = 27 Ells of Antwerp?

\[ 100 = 75L, \quad 1 = \frac{75}{100}L, \text{and} \quad 27 = \frac{27 \times 75}{100}L = 20.25L. \]

Ex. 2. If \[ \left\{ \begin{array}{l}
35 \text{ Ells of Vienna} = 24 \text{ Ells of Lyons} = 5 \text{ Ells of Antwerp} = 125 \text{ Ells of Frankfort} = 50 \text{ Ells of Frankfort}
\end{array} \right. \]

How many \[ \begin{align*}
35V & = 24L, \quad 1L = \frac{35}{24}V \\
3L & = 5A, \quad 1A = \frac{35}{24}V, \quad \text{Therefore},
\end{align*} \]

\[ 100A = 125F, \quad 1F = \frac{125 \times 35}{5 \times 24}V. \]

Ex. 3. If \[ \left\{ \begin{array}{l}
6 \text{ lb. of Pepper} \\
24 \text{ lb. of Ginger} \\
5 \text{ lb. of Cinnamon} \\
1 \text{ lb. of Sugar}
\end{array} \right. \text{ is worth} \left\{ \begin{array}{l}
12 \text{ lb. of Ginger} \\
4 \text{ lb. of Cinnamon} \\
60 \text{ lb. of Sugar} \\
6 \text{ Pence Sterling}
\end{array} \right. \]

How many \\
\text{Pence is} \quad 1 \frac{1}{2} \text{ lb. of Pepper worth?}
Since,

\[ 6P = 12G : 1G = \frac{6}{12}P \]
\[ 24G = 4G : 1G = \frac{4}{24}S \]
\[ 5C = 60S : 1C = \frac{60}{5}S = \frac{24 \times 6}{12 \times 4} \cdot \frac{24 \times 6 \times 5}{P} \]
\[ 1S = 6d = \frac{24 \times 6 \times 5}{12 \times 4 \times 50} \cdot \frac{12 \times 4 \times 60 \times 6}{24 \times 6 \times 5} \cdot \frac{112}{d} \]

And \[ 112P = \frac{12 \times 4 \times 60 \times 6 \times 112}{24 \times 6 \times 5} \]

That is, 112 lb. of Pepper is worth 111. 4s. Sterling.

V. The Rule of Alligation.

This we shall wholly omit as Imperfect, because the Questions falling under it are Indetermined ones, which this Rule, as commonly delivered, cannot fully solve; tho' it may perhaps give one or more True Answers, yet they may not be those that are required for present occasion; of which several Instances might easily be given, as the Learned Dr. Wallis has done in Ch. 52. Vol. 2. of his Works.

But as to the Solutions of such Questions, we refer the Reader to the Method already given for Solving Indetermined Questions, of which the Rule of Alligation is but an Example.

VI. The Rule of False Position.


Let \( A \) represent the Absolute Number given;
\( P \) any Supposed Number;

\( S, \) The
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The sum produced by $P$, when order'd according to the conditions of the question:

$$\frac{PA}{S}$$

is the number sought.

This is evident from the operation itself.

2. Double Position.

Let $P, p,$ be the two positions;

$$E, e, \{ \text{Their respective errors, when order'd according to the question;} \}

n \text{ The number sought: Then,}

$$n = \frac{P e \circ \circ p E}{E \circ e}; \quad \text{If } \pm E, \pm e, \quad n = \frac{P e + p E}{E + e}.

For suppose it were required to know what quantity in $a$ would give $an$.

1. If $P = n - x \quad p = n - x$

Then $n a - x a$ is not $n a - x a$.

Therefore, $E = n a - x a - x a = e = n a - n a - x a$.

And by Subduction, because the signs are alike,

$$x a \circ x a = E \circ e$$

But $P e = n x a - x a$, and $p E = n x a - x a$.

Therefore, by Subduction also,

$$P e \circ p E = (n x a \circ n x a) = n x \circ e$$

Consequently $n = \frac{P e \circ p E}{E \circ e}$, by Division.

2. If $P = n + x \quad p = n - x$

Then $n a + x a$ is not $n a$, $n a - x a$ is not $n a$.

Therefore, $E = n a + x a - n a - x a = e = n a - n a - x a$.

And by Addition, because the signs are unlike,

$$x a + x a = E + e$$

But
III. OF QUADRATIC EQUATIONS.

1. If \( xx - ax \) \( \frac{+}{-3} \) \( b = 0 \), or \( xxx = ax \) \( \frac{+}{-3} \) \( b \); put \( y = \frac{-1}{6} a = x \).  

Then \( yy - ay + \frac{1}{6} a = -ay + \frac{1}{6} a = \frac{+}{-3} b \).  

Theref. \( yy = \frac{1}{6} a a \frac{+}{-3} b \), and \( y = \frac{1}{6} a a \frac{+}{-3} b \).  

Conseq. \( x = \frac{1}{6} a \pm \frac{1}{6} a a \frac{+}{-3} b \).  

2. If \( xx + ax \) \( \frac{+}{-3} b = 0 \), or \( xxx = -ax \) \( \frac{+}{-3} b \); put \( y = \frac{-1}{6} a = x \).  

Then \( yy - ay - \frac{1}{6} a = ay + \frac{1}{6} a = \frac{+}{-3} b \).  

Theref. \( yy = \frac{1}{6} a a \frac{+}{-3} b \), and \( y = \frac{1}{6} a a \frac{+}{-3} b \).  

Conseq. \( x = -\frac{1}{6} a \pm \frac{1}{6} a a \frac{+}{-3} b \).  

Therefore if \( V \) be put for the Sign of any Term, and \( \wedge \) for the contrary, all Forms of Quadratics, with their Solutions, will be reduc’d to this one.

\[
\text{If } xx \lor ax \lor b = 0, \text{ then } = \wedge \frac{1}{6} a \pm \frac{1}{6} a a \lor b \frac{1}{6}.
\]

These Solutions may also be found thus;

\[
\begin{align*}
\text{Form } & 1. \quad rr \lor s r = -a \\ & 2. \quad rr \lor d r = +a.
\end{align*}
\]

Since
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Since \( \frac{1}{3} s \pm \frac{1}{3} d = \sum e^2 \), Therefore,

\[ \frac{1}{3} s \pm \frac{1}{3} d \times \frac{1}{3} s - \frac{1}{3} d = \frac{1}{3} s^2 - \frac{1}{3} d^2 = e, \]

And \( 4 \ e = s^2 - d^2 \), Therefore,

\[ d = s^2 - 4 \ e \frac{1}{2}, \text{and } s = d^2 + 4 \ e \frac{1}{2} \]

But \( \frac{1}{3} d^2 \frac{1}{2} = \frac{1}{3} d - \frac{1}{3} e \), and

\[ \frac{1}{3} s^2 \frac{1}{2} = \frac{1}{3} s - \frac{1}{3} e \]

Therefore:

\[ \begin{align*}
1. & \quad + a, + e = + \frac{1}{3} s \pm \frac{1}{3} s - a \frac{1}{2} \\
2. & \quad - a, - e = - \frac{1}{3} s \pm \frac{1}{3} s - a \frac{1}{2} \\
3. & \quad + a, - e = + \frac{1}{3} d \pm \frac{1}{3} d + a \frac{1}{2} \\
4. & \quad - a, + e = - \frac{1}{3} d \pm \frac{1}{3} d + a \frac{1}{2} \\
\end{align*} \]

EXAM P L E S.

1. If \( xx - 4 x = 12 \); then \( x = + 6 \), or \( - 2 \).
2. If \( xx - 2 x = 7 \); then \( x = 1 \pm 8 \), or \( 1 \pm 2 \sqrt{2} \).
3. If \( x^2 - a x + x = b - e - d \); Then,

\[ x = \frac{1}{3} a - \frac{1}{3} \pm \frac{1}{3} a^2 - \frac{1}{3} a + \frac{1}{3} + b - e - d \frac{1}{2} \]

4. If \( x^2 + \frac{a}{b} x = c \), then;

\[ x = \frac{\pm \frac{a}{4 b} b + c}{\frac{1}{2}} - \frac{a}{b} = \frac{\pm a^2 + 4 b^2 c}{4 b} - \frac{a}{b} \]

5. \( x^2 - a b x + c d x + c f x - g b = \frac{1}{4} (m + p q r - s s w) \)

Then \( x = \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} \pm \frac{1}{2} (m + p q r - s s w) \)

6. \( x^4 - a x^2 = b \), let \( x^2 = y \), then \( y^2 - a y = b \), and

\[ y = \frac{1}{2} a \pm \frac{1}{4} a^2 + b \frac{1}{2} : \quad x = \frac{1}{2} a \pm \frac{1}{4} a a + b \frac{1}{2} \frac{1}{2}. \]

7. \( x^6 = \)
7. \( x^2 - ax = b \), let \( x^2 = y \), then \( y^2 - ay = b \), and
\[
y = \frac{1}{2} \left[ a \pm \frac{a^2}{4} - 4b \right]^{\frac{1}{2}} = \frac{1}{2} a \pm \frac{1}{2} a + b \left[ \frac{1}{2} - \frac{1}{2} a \right]^{\frac{1}{2}}.
\]
8. \( nx^2 - ax = b \), then
\[
x = \frac{x}{n} + \frac{b}{n^2} \cdot x = \frac{1}{n} a \pm \frac{1}{n} a + \frac{b}{n} \left[ \frac{1}{2} - \frac{1}{2} a \right]^{\frac{1}{2}}.
\]
That is, \( n \cdot x = \frac{1}{n} a \pm \frac{1}{n} a + b \left[ \frac{1}{2} - \frac{1}{2} a \right]^{\frac{1}{2}}, \) there is \( x = \frac{1}{n} a \).

9. \( nx^2 + 2ax = c \), then \( x = \frac{2a b \pm a^2 - 4n cd}{n} \).

10. \( n^2 x^2 - m^2 x^2 - a^2 x = -c^2 \), then
\[
x = \frac{1}{n} a \pm \frac{1}{n} a - c^2 + m c \left[ 1 \right]^{\frac{1}{2}}.
\]

11. \( 5x^2 - 4x = 3 \), then \( x = \frac{2 \pm 12}{5} \).

12. \( n x^4 + dx^2 - c x^2 = a^6 - b c + c e \), let \( d = e \).

Then \( x = -\frac{1}{n} a \pm \frac{1}{n} a + b \left[ \frac{1}{2} - \frac{1}{2} a \right]^{\frac{1}{2}} \).

**Quest. 1.** A Man bought a Horse, which he sold again for 24 l. and found he had gain'd as much per Cent. as the Horse cost him; I demand what the Horse cost at first. Suppose it cost \( y \) Pounds.

Let \( 24 = a, 100 = b \),

Then \( y : a - y : b : \frac{b}{y} = y \), by the Quest.

Therefore
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Therefore \( yy + by = ba \), by Mult. and Transp.

Conseq. \( y = \frac{ba + \frac{1}{2} bb}{1} = \frac{1}{2} b \).

Or, \( y = \frac{2400 + 2500}{2} - 50 = 200 \) f. fought.

Quest. 2. A Vinnen bought a parcel of Clarret and Rhe-
nish Wine, which together cost him 120l. as for the sepa-
rate prices, 'tis only known that their Product was equal to 2700l. Required what each cost?

The Clarret cost \( c \).

Then Rheinish cost \( 120 - c \).

And \( 120c - c = 2700 \) by the Question.

Therefor. \( c = \frac{120 \times 120 - 2700}{4} \).

That is, The Clarret cost 90l. Rheinish 30l.

Quest. 3. Having bought a quantity of Goods, viz. Pepper Cloves, Ambregreice, their prices were forgot, but 'tis known that they were continually Proportional, and that the Pepper and Cloves cost 10l. the Ambregreice 24l. more than the Cloves; 'tis required what each cost.

The Pepper cost \( p \), then Cloves cost 10 - \( p \).

Then \( p : 10 - p :: 10 - p : \frac{100 - 20 p + pp}{p} \) by the Que.

And \( \frac{100 - 20 p + pp}{p} - 10 = \frac{100 - 30 p + 2pp}{p} = 24 \) by Mult. Transp. and Divis.

Therefore \( pp = 27p = 50 \), by Mult. Transp. and Divis.

And \( p = \frac{27}{2} - \frac{729 - 200}{4} = 2 \).

Therefore Pepper cost 21l. Cloves 8l. Ambregreice 32l.

III. OF
III. OF CUBIC EQUATIONS.

All Cubic Equations, whose second Term is destroyed, may be reduced into these Forms \( xxx \pm ax - b = 0 \), That is, \( xxx \pm ax = \pm b \).

Because in those where the Absolute known Quantity is Negative, there needs no more than making the Roots which were Affirmative in those, Negative in these.

If \( x^1 \pm a x - b = 0 \), then \( x^1 = a - \frac{b}{x} = 0 \).

And since the \( \left\{ \frac{\text{Diff.}}{\text{Sum}} \right\} \) of any two like Cubes, is exactly divisible by the \( \left\{ \frac{\text{Diff.}}{\text{Sum}} \right\} \) of their Roots; Therefore;

Suppose, \( b = m^3 \mp n^3 \), and \( x = m \mp n \), then the Equation will be \( \mp 3 m n \pm a = 0 \), therefore \( n = \frac{a}{3 m} \).

And \( m^9 \mp (n^3 = \frac{a^3}{27 m^3} = b \), then \( m^6 - b m^3 = \pm \frac{1}{27} a^3 \).

Therefore \( m^3 = \frac{1}{2} b + \frac{\sqrt[3]{a^3}}{2} \), but \( m^3 \pm n^3 = b \).

Then \( b = \frac{1}{2} b + \frac{\sqrt[3]{a^3}}{2} \), and \( m = n \), i.e.

\[
x = \sqrt[3]{b} + \sqrt[3]{\frac{b^2}{4} + \frac{a^3}{27}} = \sqrt[3]{-\frac{b}{2} + \frac{\sqrt[3]{b^2 + a^3}}{4}} + \frac{\sqrt[3]{b^2 + a^3}}{27}
\]

Therefore,

1. If \( xxx \mp a x \pm b = 0 \), Then,

\[
x = \sqrt[3]{b} + \sqrt[3]{\frac{b^2}{4} + \frac{a^3}{27}} - \sqrt[3]{-\frac{b}{2} + \frac{\sqrt[3]{b^2 + a^3}}{4}} + \frac{\sqrt[3]{b^2 + a^3}}{27}
\]

2. If...
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2. If \( a x + b = 0 \), Then

\[
x = -\frac{\sqrt{\frac{3}{2} b^2 + \frac{3}{4} b^2 - \frac{a^2}{27}}} + \sqrt{\frac{3}{2} b - \sqrt{\frac{3}{4} b^2 - \frac{a^2}{27}}}.
\]

Having discovered one of the Roots of an Equation, the rest may be found by Division.

And as for Equations of higher Dimensions, we shall omit the like Methods that might be given for their Solutions: because more intricate and tedious for Practice, than the Rules of Numerical Exegesis delivered by Vieta, Harriot, and Oughtred; which are yet abundantly improved by the Method of Infinitely Converging Series, (an Universal way of Extracting the Roots of any Equation whatsoever,) And of this Method we shall in its proper Place, give General Examples, whence all possible Particular ones may be drawn.

NOTE.

That Equations of high Dimensions may oftentimes be advantageously reduced lower, by some of the former Rules, or those that may easily be deduced from them; according to the Methods of Hudde, Merrie, and others; and then the Solution will be less troublesome.
CHAP. VIII.

Of Arithmetic Progression.

DEFINITION I.

A Continued Arithmetic Proportion, that is, where the Terms do Increase and Decrease by equal Differences, is called Arithmetic Progression.

Thus \(\{a, a+d, a+2d, a+3d, \ldots\}\) Increasing \(\{\) by \(d\).

\(\{a, a-d, a-2d, a-3d, \ldots\}\) Decreasing \(\{\)

SCHOLIUM 1.

But since this Progression is only a Compound of two Series,

\(\{\) Equals \(\{a, a, a, a, a, \ldots\}\) \(\{Arith. Prop.\) \(\pm d, \pm 2d, \pm 3d, \pm 4d, \ldots\) \(\{\).

Therefore the most Natural Arithmetic Progression is that which begins with \(0\);

As \(0, \pm d, \pm 2d, \pm 3d, \pm 4d, \pm 5d, \ldots\) Increasing \(\{\) Decreasing.

SCHOLIUM 2.

In an Arithmetic Progression;

\[\{a \atop n \text{ terms}\} \]

If \(n\) be the First Term

Common Difference

Number of Terms

Last Term

Sum of all the Terms

Then
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Then any *Three* of the *ele Terms* being *given*, the other *Two* are easily *found*.

And the several *Cases* are reducible into *Ten Propositions*, which are all *Solved* by the *Two* following *Lemmata*.

**LEMMA 1.**

In any *Arithmetic Progression*:

\[ \frac{n}{2} : : a + l : s. \]

For

\[
\begin{align*}
\left( \frac{a}{a+2d} \right) & + \left( \frac{1-2d}{1+2d} \right) = \left( \frac{a+l}{a+l} \right) \\
\left( \frac{a+3d}{1+3d} \right) & + \left( \frac{1-3d}{1+3d} \right) = \left( \frac{a+l}{a+l} \right)
\end{align*}
\]

Therefore

\[ s + s = \frac{a+l}{a+l} \times n. \]

That is,

\[ 2s = \frac{a+l}{a+l} \times n. \]

Consequently,

\[ \frac{n}{2} : : a + l : s. \]

**COROLLARIES.**

1. \[ a = \frac{2s}{n} - l = \frac{2s-nl}{n} = 2s - nl \times \frac{1}{n} \]

2. \[ n = \frac{2s}{a+l} = \frac{2s}{a+l} = \frac{s}{a+l} \times \frac{1}{a+l} \]

3. \[ l = \frac{2s}{n} - a = \frac{2s-na}{n} = 2s - na \times \frac{1}{n} \]

4. \[ s = \frac{n}{2} \times \frac{a+l}{2} = \frac{n}{2} \times \frac{a+l}{2} = n \times \frac{a+l}{2} \]

\[ X 2 \]

**LEM-**
LEMMA 2.

In any Arithmetic Progression;

Tis \( n - 1 \):
\[ d : l - a. \]

For, \( a, a + d, a + 2d, a + 3d, \ldots \) x \( d = l \).

That is, \( n - 1 \times d = l - a \), by Transpos.

Therefore \( n - 1 :: d : l - a \).

COROLLARIES.

1. \( a = \frac{l - n}{-1} - 1 \times d = l - n d + d \).
2. \( n = \frac{l - a}{d} + 1 = \frac{l - a + d}{d} \).
3. \( d = \frac{l - a}{n - 1} = l - a \times \frac{1}{n - 1} \).
4. \( l = a + n - 1 \times d = a + nd - d \).

PROPOSITION 1.

Given \( a, d, n \); Required \( l, s \).

SOLUTION.

1. \( l = a + nd - d = \frac{2s - na}{n} \) by Lem. 2d. and 1st.

Then \( na + nud - nd = 2s - na \), by Mult.

And \( 2s = 2na + nnd - nd \), by Transp.

2. Therefore \( s = \frac{na + nud - nd}{2} \) by Division.

P. R. Q.
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PROPOSITION 2.
Given, \(a, d, l\); Required, \(n, s\).

SOLUTION.
1. \(n = \frac{l - a + d}{d} = \frac{2s}{a + l}\) by Lem. 2d. and 1st.

Then \(2ds = ll + ld - a^2 + ad\), by Mult.

2. Therefor, \(s = \frac{ll + ld - a^2 + ad}{2d}\), by Division.

PROPOSITION 3.
Given, \(a, d, s\); Required \(n, l\).

SOLUTION.

Since \(l = \frac{2s - na}{n} = a + nd - d\), by Lem. 1. and 2.

Therefor, \(n + 2na - nd = 2s\) by Mult. and Transp.

And \(n + \frac{2a - d}{d}n = \frac{2s}{d}\) by Division.

1. Then \(n = \frac{da + \frac{1}{2}dd - ad + 2ds}{d}\) - \(a + \frac{1}{2}\).

And because \(n = \frac{2s}{a + l} = \frac{l - a + d}{d}\) by Lem. 1. and 2.

Therefor, \(ll + dl = 2ds - ad + aa\), by Mult. and Transp.

2. Then \(l = 2ds - ad + aa + \frac{1}{2}dd - \frac{1}{2}d\), by Completing the Square, and Evolvs.
PROPOSITION 4.
Given, \( a, l, s; \) Required \( n, d. \)

SOLUTION.

1. \( u = \frac{2s}{l+a} = \frac{l-a+d}{d} \) by Lem. 1 and 2.
Then \( 2ds-ld-ad = ll-a \) by Mult. and Trans.
2. Therefore, \( d = \frac{ll-a}{2s-l-a} \) by Division.

PROPOSITION 5.
Given \( a, n, s; \) Required \( l, d. \)

SOLUTION.

1. \( l = \frac{2s-na}{n} = a + nd - d \) by Lem. 1. and 2.
Then \( nnnd - nd = 2s - 2na \) by Mult. and Trans.
2. Therefore, \( d = \frac{2s-2na}{nn-n} \) by Division.

PROPOSITION 6.
Given \( a, n, l; \) Required \( d, s. \)

SOLUTION.

1. \( d = \frac{l-a}{n-1} = \frac{l-a \times \frac{1}{n-1}} \) by Lemma 2.
2. \( s = \frac{na+nl}{2} = a + \frac{l \times n}{2} \) by Lemma 1.
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PROPOSITION 7.
Given d, l n; Required a, s.

SOLUTION.

1. \( a = l - nd + d = \frac{2s - nl}{n} \) by Lem. 2. and 1.

Then \( 2s = 2nl - nnd + nd \), by Mult. and Trans.

2. Theref. \( s = \frac{2nl - nnd + nd}{2} \) by Division.

PROPOSITION 8.
Given d, n, s; Required a, l.

SOLUTION.

Since \( l = a + nd - d = \frac{2s - na}{n} \), by Lem. 2. and 1.

Then \( 2na = 2s - nnd + nd \), by Mult. and Trans.

1. Theref. \( a = \frac{2s - nnd + nd}{2n} \), by Division.

And since \( a = l - nd + d = \frac{2s - nl}{n} \), by Lem. 2. and 1.

Then \( 2nl = 2s - nnd - nd \), by Mult. and Trans.

2. Theref. \( l = \frac{2s + nnd - nd}{2n} \), by Division.
PROPOSITION 9.
Given, \( d, l, s \); Required \( a, n \).

SOLUTION:

Since \( n = \frac{2s}{a + l} = \frac{l - a + d}{d} \), by Lem. 1. and 2.
Then \( a = \frac{l + l + 1d - 2 ds}{2 s + \frac{1}{2} d} \).

1. Thereof. \( a = \frac{l + l + 1d - 2 ds}{2 s + \frac{1}{2} d} \).
And because, \( a = \frac{l - n d + d}{n} \), by Lem. 2 & 1.

Thereof. \( n d + 2 n l + n d = 2 s \), by Mult. & Trans.
And \( n d + 2 n l + n d = 2 s \).

2. Then \( n = \frac{l + l + 1d + 1d - 2 ds}{d} \).

PROPOSITION 10.
Given, \( n, l, s \); Required, \( a, d \).

SOLUTION.

1. \( a = \frac{2s - nl}{n} = l - nd + d \) by Lem. 1. and 2.
Then \( 2 n l - 2 s = n d - nd \), by Mult. and Trans.

2. Thereof. \( d = \frac{2n l - 2s}{nn - l} \) by Division.
PROBLEM I.

To find the Sum of the Powers of any Arithmetic Progression.

PREPARATION.

Suppose \( n \) the Index of the Power.

Let each Term of the Progression be raised to each Power, under that whole Sum is sought.

And let the Sum of each Rank so raised be multiplied by the Multiple of the like Dimension of \( a \) in \( a + d, a + 2d, \ldots, a + (n-1)d \).

Put \( S \) for the Sum of all the Products.

And \( m \) for the Multiple of \( a^n \) in the Power \( a + d, a + 2d, \ldots, a + (n-1)d \).

SOLUTION.

Then

\[
\frac{\frac{1}{m} \cdot a + d + \cdots + \frac{1}{m} \cdot a + (n-1)d}{\frac{1}{m} \cdot (n-1)} = \frac{a + d + \cdots + a + (n-1)d}{m} = \frac{a + d + \cdots + a + (n-1)d}{m}
\]

is the Sum of any Series of Powers whose Roots are Arithmetically Proportional.

For Suppose the Sum of the Cubes of this Arithmetic Progression, \( a, a + d, a + 2d, \ldots, a + 3d \), was required.

1. \( a + d + \cdots + a + (n-1)d = a^3 + 4a^3d + 6a^3d^2 + 4a^3d^3 + 6a^3d^4 + a^3d^5 \)

And the Sum of this Series is \( 4a^3 + 6d \).

Which Multiply by \( 4d^3 \) (the Multiple of \( a \) in \( a + d, a + 2d, \ldots, a + 3d \))

The Product will be \( 16a^3d^3 + 24a^3d^4 \)

Also the Sum of their Squares is \( 4a^6 + 12ad^2 + 14d^2 \)

Which Multiply by \( 6d^2 \) (the Multiple of \( a^2 \) in \( a + d, a + 2d, \ldots, a + 3d \))

The Product will be \( 24a^5d^2 + 72ad^8 + 84d^6 \)

\( \therefore \) Therefore
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Synopsis

Part I. Sect. 3.

There. \( z = 24a^3d^3 + 88a^1d^1 + 108d^4 = \text{Sum of these Prod.} \)

To which add \( a^4 + 4d^4 \)

\[ = a^{n+1} + nd^{n+1} + z \]

Sum is \( a^4 + 24a^3d^3 + 88a^1d^1 + 112d^4 \)

From \( I + d^{n+1} - a^{n+1} + nd^{n+1} + z = a^4 + 16a^3d + 96a^1d^1 + 256d^4 \)

Subd. \( a^{n+1} + nd^{n+1} + z = a^4 \)

\[ \quad + 24a^3d + 88a^1d^1 + 112d^4 \]

Then

\[ I + d^{n+1} - a^{n+1} + nd^{n+1} + z = 16a^3d + 72a^1d^1 + 168a^1d^1 + 144d^4 \]

And \( \frac{16a^3d + 72a^1d^1 + 168a^1d^1 + 144d^4}{4d} = 4a^3 + 18a^1d + 42a^1d^1 + 36d^3 \), the Sum of the Cubes of the given Terms.

Because,

\[ \begin{align*}
    &a^3 \\
    &a^2d \\
    &a^1d^2 \\
    &a^0d^3
\end{align*} \]

The Cube of

\[ \begin{align*}
    &a^3 + 3a^2d + 3a^1d^1 + d^3 \\
    &a^2d + 6a^1d^1 + 12a^1d^1 + 8d^3 \\
    &a^1d^2 + 9a^1d^1 + 27a^1d^1 + 27d^3
\end{align*} \]

The Sum is \( 4a^3 + 18a^1d + 42a^1d^1 + 36d^3 \),

the same with the Quotients found.

'Tis the same in any other Series for any other Power.

COROLLARY 1.

Therefore, In a Series of Laterals beginning with \( r \),

if \( s \) be put for the Sum of the \( s \) Power thereof, Then,

\[ \frac{1}{1 + r} = \frac{1}{1 + n + r} = m \]

or,

\[ \begin{align*}
    1. & \quad I + 1^2 = 2s \\
    2. & \quad I + 1^3 = 3s \\
    3. & \quad I + 1^4 = 4s \\
    4. & \quad I + 1^5 = 5s' \\
    5. & \quad I + 1^6 = 6s'
\end{align*} \]

&c.

COROLL.
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COROLLARY 2.

Hence, when \( a = 0 \), then \( n = l + 1 \), and

1. \( \frac{l+1}{2} - \frac{l}{2} = \frac{n}{2} \)

2. \( \frac{l+1}{2} + \frac{l+1}{2} = \frac{2l+2}{2} = 2^l \)

3. \( \frac{l+1}{2} \cdot \frac{l+1}{2} + \frac{l+1}{2} = \frac{2l+3l+1}{2} = 3^l \)

&c.

Therefore,

1. \( s = \frac{l+1}{2} = \frac{l+1}{2} = \frac{n}{2} \)

2. \( s = \frac{2l+3l+1}{6} = \frac{l+l+1}{6} + \frac{l+1}{6} = \frac{n}{3} + \frac{n}{6} + \frac{n}{6} \)

3. \( s = \frac{l^2 + 2l^2 + l^2}{4} = \frac{l+l}{4} + \frac{l+1}{4} = \frac{n^2}{4} + \frac{n^2}{4} \)

&c.

COROLLARY 3.

But if the Number of Terms \( n \) be supposed Infinite, and \( g \) be put for the greatest Term or Power;

Then \( s = \frac{ng}{n+1} \), therefore \( l : l + 1 : : s : ng \).

This proportion will hold, whether \( n \) be Affirmative, or Negative, Whole, Fraction, or Surd Quantity.

DEFINITION II.

The Sums of Numbers in a Continued Arithmetic Proportion from Unity are called Figurate or Combinatory Numbers. Thus,
If \( 1, 1+d, 1+2d, 1+3d, 1+4d, 1+5d, 1+6d, \ldots \) be an **Arithmetic Progression**; then
\[ 1, 1+1+d, 1+1+d+1+2d, 1+1+d+1+2d+1+3d \ldots \]
are **Figurate Numbers**.

For the **Units** in each may be disposed into the **Form of a Regular Polygon**, whose **Number of Angles** is \( d+2 \), and whose **Side** \((s)\) is equal to the **Number of Terms** that compose it.

Let \( f \) be the **Figurate Number**, a **Side**, or **Number of Terms** composing it.

\[ a = \text{the Number of Angles} \left( = d + 2 \right). \]

**Problem 2.**

**Case 1.** Given \( s, d \); Required \( f \)?

**Solution.**

Since \( f \) is but the **Sum of a Series Arithmetically Proportional** beginning with \( 1 \), by this **Definition**;

Therefore

\[
\frac{s+s}{2} \quad (= \frac{s \times 1+1}{2}) = f, \quad \text{by Lemma 1.}
\]

But

\[ s \quad d = d + 1 = l \quad \text{by Lemma 2.} \]

Consequently

\[
f = \frac{2s + ssd - sd}{2} = s + \frac{ss - sx d}{2} \quad \text{by Subs.}
\]

**Case 2.** Given \( s, a \); Required \( f \)?

**Solution.**

Since \( s = d + 2 \), or \( d = s - 2 \), by this **Definition**;

Therefore

\[
f = \frac{4+sx-a-2s \times s}{2} \quad \text{by Substitution.}
\]
Problem 3.
Case 1. Given \( f, d \); Required \( s \).

SOLUTION.

Since \( f = \frac{2s - ssd - sd}{2} = s + \frac{ssd - sd}{2} \) by Prob. 2,

therefore,

\[
2s - sd = ss + s \times \left( \frac{2 - d}{d} \right) = \frac{2f}{d},
\]

Conseq.,

\[
s = \frac{3df + dd - 4d + 4f + d - 2}{2d}
\]

Case 2. Given \( f, a \); Required \( s \).

SOLUTION.

Since \( d = a - 2 \), by this Definition.

therefore,

\[
s = \frac{a^2 - 16a + 16 + 8af - 15f + a - 4}{2a - 4}
\]

Definition III.

The \{ Polygonals \}

\{ 1st. Pyramidals \} from Unity \{ 1st. \}

\{ 2d. Pyramidals \} are called \{ 2d. \}

\{ 3d. Pyramidals \} &c. \{ 3d. \}

Pyramidals, having their Names from their Number of Sides.

Corollary.

Therefore in a Rank of

\{ Units \}

\{ Lateralis \}

\{ Triangulars \}

\{ 1st. Pyramid \}

\&c.

\{ their Sums are called \}

\{ Lateralis \}

\{ Triangulars \}

\{ 1st. Pyramid \}

\&c.

\{ the \}

\{ 2d \}

\{ 3d \}

\{ 4th. \}

\&c.

Order.

\{ 2d \}

\{ 3rd \}

\{ 4th. \}

\&c.
EX AM P L E.

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</tbody>
</table>

1. Here 'tis evident, that each Figurate Number is the Aggregate of the preceding Series so far.

2. That each Figurate Number is also equal to the Sum of the preceding one, and that above it.

P R O B L E M 4.

To find the Sum of such Series, or to find any particular Figurate Number, by having the Side or Number of Terms composing it given.

1. In Triangulars or Figurates of the 3d. Order.

Since every Triangular Number is \( \frac{s(s+1)}{2} \), by Definis.3.

Theref. \( \frac{A + A}{2} = \frac{A + A}{2} = \frac{s + \frac{s}{2}}{2} \), &c. is a Series of Triangulars, or Figurates of the 3d. Order.

But \( A + A + A \), &c. = \( \frac{s + \frac{s}{2}}{2} \)

And \( A + A + A + A \), &c. = \( \frac{s + \frac{s}{2} + \frac{s + s}{2}}{3} \)

Theref.
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Theref. \[ \frac{AA + AA + AA + A + a \&c.}{2} = \frac{s^3 + 3s^2 + 2s}{6} \]

is the Sum of any Series of Figurates of the 3d. Order, or a Figurate Number of the 4th Order, whose side is s.

II. In Figurate Numbers of the 4th Order.

Since every Figurate of the 4th Order is \( \frac{s^3 + 3s^2 + 2s}{6} \) by the preceding Problem; Therefore the Series is

\[ \frac{A^3 + 3A^2 + 2A}{6}, \quad \frac{A^3 + 3A^2 + 2A}{6}, \quad \frac{a^3 + 3a^2 + 2a}{6} \&c. \]

But

\[ \begin{cases} 2A + 2A + 2a, \quad \&c. = ss + s \\ 3A^2 + 3A^2 + 3a^2, \quad \&c. = s^3 + s^3 + \frac{s^3 + s^3}{2s} \\ A^3 + A^3 + a^3, \quad \&c. = \frac{s^4 + s^4 + s^4 + s^3}{4} + \frac{s^4 + s^4 + s^4 + s^3}{4} \end{cases} \]

by Cor. 2. Prob. 1.

And \( \frac{1}{2} \) of the Sum must be \( \frac{s^4 + 6s^3 + 11s^2 + 6s}{24} \) which is the Sum of a Series of Figurates of the 4th Order, or a Figurate of the 5th Order, whose side is s.

Therefore,

\[ \begin{cases} \frac{s^2 + s}{2} = \frac{s + 0}{1} x \frac{s + 1}{2} \\ \frac{s^2 + 3s^2 + 2s}{6} = \frac{s + 0}{1} x \frac{s + 1}{2} x \frac{s + 2}{3} \\ \frac{s^4 + 6s^3 + 11s^2 + 6s}{24} = \frac{s + 0}{1} x \frac{s + 1}{2} x \frac{s + 2}{3} x \frac{s + 3}{4} \end{cases} \&c. \]

is a Figurate \( \{ 3d. \} \) Order of the Sum of \( \{ 2d. \} \) \( \{ 3d. \} \) Order. Number of \( \{ 4th. \} \) a Series of Figurates \( \{ 5th. \} \) of the \( \{ 4th. \} \) Order.

\[ E \ X. \]
EXAM P L E.

Given, The Number of Terms, suppose 8, in a Series of Figurates of the 4th Order;

Required, The Sum of that Series; or the 8th Figurate of the 5th Order.

Substitute 8 in the room of s, then by the Rule,

\[ \frac{8-1}{1} \times \frac{8-2}{3} \times \frac{8-3}{4} = 330, \text{ required.} \]

S C H O L I U M.

From what has been here said, we may easily investigate that excellent Theorem of the Illustrious Mr. New- ton, for Raising a Binomial, to any given Power.

For let any Binomial \((a+x)\) be raised to any Power, whose Index suppose \(n\), (representing any Number, Affirmative or Negative, Integer or Fraction); The several Powers of that Binomial are,

1. \(a + 1x\)
2. \(a^2 + 2ax + 1x^2\)
3. \(a^3 + 3a^2x + 3ax^2 + 1x^3\)
4. \(a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + 1x^4\)
5. \(a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + 1x^5\)
6. \(a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + 1x^6\)
&c.

Here 'tis evident at sight, that the Union of

\[ \begin{align*}
1\text{st. Term is } &a^n, \\
2\text{d. Figurate } &\text{Order, whose Place } n-1, \\
3\text{d. Number } &\text{Or Side is express } n-2, \\
4\text{d. } &\text{of the } \text{Side } n-2,
\end{align*} \]

&c.

Which
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Which being Substituted in the room of $s$, in the preceding Theorems (for finding such Figurate Numbers) shall give.

\[
\begin{align*}
1 & \quad n \quad 0 \\
1x & \quad 1 \\
1x^2 & \quad 2 \\
1x^3 & \quad 3 \\
\vdots & \\
1x^n & \quad n \\
\end{align*}
\]

\[
\begin{align*}
1 & \quad n \quad 0 \\
1x & \quad 1 \\
1x^2 & \quad 2 \\
1x^3 & \quad 3 \\
\vdots & \\
1x^n & \quad n \\
\end{align*}
\]

OR

The Uncie or Coefficients of the 1st, 2d, 3d, 4th, &c. Term.

Therefore, the Uncie of any Binomial $(a + x)$ rais'd to the Power whose Index is $n$, will be

\[
\begin{align*}
1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6}, & \text{ &c. i.e.} \\
1x^0 \times 1x^1 \times 1x^2 \times 1x^3, & \text{ &c.}
\end{align*}
\]

respectively. Therefore,

\[
(a + x)^n = a^n + \frac{n}{1} a^{n-1} x + \frac{n}{1} \frac{n-1}{2} a^{n-2} x^2 + \frac{n}{1} \frac{n-1}{2} \frac{n-2}{3} a^{n-3} x^3 + \frac{n}{1} \frac{n-1}{2} \frac{n-2}{3} \frac{n-3}{4} a^{n-4} x^4 + \&c.
\]
Or putting $q = \frac{x}{a}$, then $a + x = a + a \cdot \frac{x}{a} = a + a q = a q + f q$

And $a + a q = a ^ {n}$

$= \frac{n - 1}{1} a ^ {n - 1} q + \frac{n - 1}{2} a ^ {n - 2} q ^ {2} + \frac{n - 1}{3} a ^ {n - 3} q ^ {3} + \frac{n - 1}{4} a ^ {n - 4} q ^ {4} + \&c.$

Also putting $\text{A} = 1$, $\text{B} = 2$, $\text{C} = 3$, $\text{D} = 4$, &c., it will be,

$= a + a q = a ^ {n} + \frac{n - 1}{1} A q + \frac{n - 1}{2} B q + \frac{n - 2}{3} C q + \frac{n - 3}{4} D q + \&c.$

Or if the index be $\frac{m}{n}$, then:

$= a + a q = a ^ {n} + \frac{n}{m} A q + \frac{m - n}{2 n} B q + \frac{m - 2 n}{3 n} C q + \frac{m - 3 n}{4 n} D q + \&c.$

Note, That there are several other ways of Investigating this Important Theorem, yet none here would be so apposite as this.

And we have insisted the more therein, because its extensive Use is almost Infinite; since 'tis not for the Discovery of one Particular Case alone, that it serves, 'tis not only Involution, Evolution, Division by Powers, or Radical Quantities, and the like, as well in Numbers as Species that it performs; But it even Comprehends the Method of Indivisibles, the Arithmetic of Infinities, the Doctrine of Series: and in a word, there is scarce any Inquiry so Sublime and Intricate, or any Improvement so Eminent
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and Considerable, in *Pure Mathemes*, but by a *Prudent Application* of this *Theorem*; may easily be exhibited and deduced; and that by a *General and Direct Calculation*, in as Perfect a manner as the Nature of the *Thing* will admit; a few of which we shall Instance.

I. To raise *any Tri-nomial, Quadri-nomial, &c.* or *Infinite-nomial* to *any given Power.*

Suppose the *Infinite-nomial* $a + b \cdot x + c \cdot x^2 + d \cdot x^3$, &c. was to be raised to the *Power* whose *Index* is $n$.

Put $b \cdot x + c \cdot x^2 + d \cdot x^3$, &c. $= x^3$ in the *Binom. Theor.*

And $b \cdot x + c \cdot x^2 + d \cdot x^3$, &c. $= x^3$ &c. Then we have this *Theorem.*

\[
\begin{align*}
&= a^n + \frac{n}{1} a^{n-1} \cdot b \cdot x + \frac{n}{1} \cdot \frac{n-1}{2} a^{n-2} \cdot b^2 \cdot x^2 + \frac{n}{1} \cdot \frac{n-1}{3} a^{n-3} \cdot b^3 \cdot x^3 + \cdots
\end{align*}
\]

that is

\[
\begin{align*}
&= a^n + \frac{n}{1} a^{n-1} b \cdot x + \frac{n}{1} \cdot \frac{n-1}{2} a^{n-2} b^2 \cdot x^2 + \frac{n}{1} \cdot \frac{n-1}{3} a^{n-3} b^3 \cdot x^3 + \cdots
\end{align*}
\]
And Multiplying both Sides by $\pi^n$, 'twill be

$$\pi^2 + bx^2 + cx^3 + dx^4 \ldots \pi = a^n \pi^n$$

$$+ \frac{n}{1} a^{n-1} b \pi^{n-1}$$

$$+ \frac{n}{1} \times \frac{n-1}{2} a^{n-2} b^2 \pi^{n-2}$$

$$+ \frac{n}{1} a^{n-1} c$$

$$+ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} b^3 \pi^{n-3}$$

$$+ \frac{n}{1} \times \frac{n-1}{1} a^{n-2} b c$$

$$+ \frac{n}{1} a^{n-1} d$$

&c. as express'd by the Author thereof, the Excellent Analyst Mr. Ab. de Moivre, (in Philos. Trans. N. 230.)

Here 'tis manifest, that

1. All the Products, that can be made so as to have the Sum of the Exponents of the Letters composing them equal to some Index of the Power of $\pi$, must belong to the Power.

2. The Number, which expresses how many ways the Letter of each Product may be changed, must be prefix'd to the Product.
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But by putting $A = \text{1st Term, } B = \text{2d, } C = 3d, D = 4\text{th}$, &c., then will

$$\frac{a^2 + b^2 + c^2 + d^2 + \ldots}{a^2} \text{ be found equal to}$$

$$\frac{1}{nA} + \frac{B}{1a} = \frac{2nA + 2n - 1B}{2a}$$

$$\frac{3nA + 2m - 1Bc + 1n - 2Cb}{3a}$$

$$\frac{4nA + 3n - 1Bd + 2n - 2C + 1n - 2Db}{4a}$$

$$\frac{5nA + 4n - 1Be + 3n - 2Cd + 2n - 3De + 1n - 4Ed}{5a}$$

&c., where the Law of Continuation is visible; but the Application to Practice is somewhat more difficult than the former.

II. Of the Nature and Construction of Logarithms.

From hence also, the Celebrated Mathematician Mr. Halley, Savilian Professor of Geometry in Oxford, among the several admirable Discoveries, and happy Advances, in Useful Learning, which he obliges the World with, has (in Philos. Trans. N. 216.) drawn a very curious Method for constructing Logarithms, not only comprehending all the Improvements that Mercator, Gregory, and others have made by the Help of Geometric Figures, but shewing with great Accuracy, from the Common Properties of Numbers, (as most Natural and Agreeable in Things purely Arithmetical,) how the Logarithms may be produced to any desired Number of Places, with far more Ease and Expedition than by any Method known before. Since
Since these Logarithms (Invented by the Lord Napier) as Improved by Mr. Briggs, are one of the most useful Discoveries in Arithmetick, and the Demonstration of this Method of Constructing them follows properly in this Place, and is so worthy the Reader's Knowledge; we have here inserted a Specimen of their Nature and Construction, according to Mr. Halley's Method.

1. Supposing an Infinite Number $n$ of equal Ratio's or Ratiauncula in a continued Scale of Proportionals between the two Terms of any Ratio, as between $1$ and $1 + x$ or $1 + x^\frac{1}{n}$, then $1 + x$ will be the First Mean of Root of the Infinite Power $1 + x^\frac{1}{n}$, and let $x$ be a Ratiauncula, or Fluxion of the Ratio of $1$ to $1 + x$, then

$$\frac{1}{1 + x} \approx \frac{1}{1} + \frac{1}{1} \cdot \frac{1}{1 + x} \approx 1 + x$$

$$\frac{1 + x}{1 + x} = 1 + x$$

where 'tis evident, that any Index expresseth the Number of Ratiauncula contain'd in the Ratio of $1$ to such Term.

2. Hence, we may value Ratio's by the Number of Ratiauncula contain'd in each, and may consider them as Quantitates suae generis, beginning from the Ratio of $1$ to $1 = o$, being Affirmative, or Negative, according to their Increase or Decrease, above or below Unity.

So that Ratio's may be to one another as the Number of like and equal Ratiauncula contain'd between their Terms; and their Duplicate, Triplicate, &c. contains Twice, Three, &c. that Number.

3. And
3. And the Number of Rationals between 1 and any Number or the Value (i.e. the Exponent) of the Ratio of Unity to any Number, is called the Logarithm of that Number, as it's E להםור indicates very properly Imports.

Thus suppose between 1 and 10 an Infinum Number of mean Proportionals, expressed by 10000 &c. in infinitum.

Then between 1 and 2, 1 and 3, 1 and 4, 1 and 10; there will be 3010 &c. 4771 &c. 6985 &c. 10000 &c. which are the Logarithms of 2, 3, 4, 10, or rather the Logarithms of the Ratio of 1 to 2, 1 to 3, 1 to 4, 1 to 10.

So if the Ratio of 1 to 10
That of its \{ Duplicate \ Triplet \ &c. \}
contains \{ 1000 \ 2000 \ 3000 \ &c. \} Rationals

Therefore, Logarithms or the Values of Ratio's are in an Aritmetic Progresion.

4. But because any Infinum Number of means may be taken between the Terms of any Ratio, provided the same Proportion be every where observed: therefore \( n \times \) may as well be put for the Logarithm of \( 1 \) to \( 1 + x \), i.e. the Sum of the Rationals may as well be the Index as the Number of them; then,

\[
\begin{align*}
1, x, 2x, 3x, 4x, &c. \times \times \\
1, 1+x, 1+x^2, 1+x^3, 1+x^4, &c. 1+x
\end{align*}
\]

Therefore \( 1+x \)
And \( 1+x = \times - 1 \times n = n \times = L, 1+x \)

Therefore
Theref. \( 1 + \frac{1}{1000} \times \frac{1}{x} \times 1000 \times \frac{1}{x} = L, 10 \).

And \( 1 + \frac{1}{1010} \times \frac{1}{x} \times 3010 \times \frac{1}{x} = L, 2 \).

Whence, \( L, 1 + \frac{1}{x} : L, 1 + \frac{1}{x} :: N \times \frac{1}{x} : n \times \frac{1}{x} \).

And these \textit{Ratiuncula} being hitherto considered as having the same magnitude in all \textit{Ratio's}.

Therefore \( L, 1 + \frac{1}{x} : L, 1 + \frac{1}{x} :: N : n \). That is, the \textit{Logarithms of Ratio's are as the Number of Ratiuncula contain'd between their Terms}:

But if the \textit{Number of Ratiuncula} be supposed the same in all \textit{Ratio's},

Then \( L, 1 + \frac{1}{x} : L, 1 + \frac{1}{x} :: \frac{x}{x} : \frac{n}{n} \), that is, \textit{The Logarithm of each Ratio will be as the Fluxion thereof}.

Therefore the \textit{Logarithm of any Number} is found by taking the \textit{Difference between Unity and the Infinite Root of that Number}.

Hence 'tis evident that there may be as many different \textit{Scales of Logarithms}, as there are assumed different \textit{Infinite Indices} \((n)\) of the Power whose \textit{Root} is sought.

If \( n = \frac{1}{5} \times 1000 \times \frac{1}{x} \), the Lord \textit{Naper's Logarithm} will be produced.

Therefore, put \( \frac{1}{x} \) for the \textit{Infinite Power} to be resolv'd; then,

The \textit{Coefficients} being \( 1 \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \frac{1}{6} \).

&c. when \( n \) is \textit{Finite}, by the former \textit{Rules}.
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But \[ \frac{\frac{1}{2} - \frac{1}{2}}{1} = \frac{1}{2} + \frac{1}{2} \]

\[ \frac{\frac{2}{3} - \frac{2}{3}}{1} = \frac{2}{3} + \frac{2}{3} \]

\[ \frac{\frac{3}{4} - \frac{3}{4}}{1} = \frac{3}{4} + \frac{3}{4} \]

And \( n \) being *infinity infinite*, therefore that which is divided thereby must vanish: Consequently the Coefficients will be

\[ + \frac{1}{n} - \frac{2}{n} + \frac{3}{n} - \frac{4}{n} + \frac{5}{n} \&c. \quad \text{And} \]

\[ \frac{1}{n} \times \frac{1}{n} = \frac{1}{n} + \frac{1}{3n} - \frac{1}{5n} \&c. \quad \text{Hence} \]

\[ \frac{1}{n} \times \frac{1}{n} + \frac{1}{n} \times \frac{1}{3n} + \frac{1}{n} \times \frac{1}{5n} \&c. = 1 + \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \frac{1}{4n} + \frac{1}{5n} + \&c. = L + x \]

And since the *Infinite Index* \( n \), may be assumed at pleasure, the several *Scales of Logarithms* to such Indices will be as \( \frac{1}{n} \), or Reciprocally as such Indices. Therefore, if \( n = 10000 \), \&c. as in *Napier’s Logarithms*; then

\[ x + \frac{1}{n} x^2 + \frac{1}{3n} x^3 + \frac{1}{5n} x^4 + \&c. = x = L + x \]

But if neither of the *Terms* of the *Ratio* be 1, they must be reduced into such, wherein one of the *Terms* may be 1: Thus,

If the Least be \( a \), the Greatest \( b \); let \( s = b - a, d = b - a \);

\[ \begin{aligned}
  a : b :: 1 : 1 + x = \frac{b}{a} & \therefore x = \frac{b}{a} - 1 = \frac{b - a}{a} = \frac{d}{a} \\
  b : d :: 1 : 1 - x = \frac{a}{b} & \therefore x = \frac{a}{b} - 1 = \frac{b - a}{b} = \frac{d}{b} \\
  A = 2 \\
  \text{Therefore} \\
\end{aligned} \]
Therefore the Logarithm of the Ratio contain'd between \( a \) and \( b \), may be doubly express'd.

Now suppose the Ratio of \( a \) to \( b \), was divided into that of \( a \) to \( \frac{1}{2} s \), and of \( \frac{1}{2} s \) to \( b \).

Then \( \frac{a}{\frac{1}{2} s} + \frac{b}{b} = \frac{a}{\frac{1}{2} s} \), for \( \frac{a}{\frac{1}{2} s} \times \frac{b}{b} = \frac{a}{s b} = \frac{a}{b} \)

Or \( \frac{\frac{1}{2} s}{a} + \frac{b}{\frac{1}{2} s} = \frac{a}{\frac{1}{2} s} \), for \( \frac{\frac{1}{2} s}{a} \times \frac{b}{\frac{1}{2} s} = \frac{b}{s a} = \frac{b}{a} \)

And

\[
\begin{cases}
\frac{\frac{1}{2} s}{a} : a : 1 : 1 - x = \frac{a}{\frac{1}{2} s} \\
\frac{\frac{1}{2} s}{b} : b : 1 : 1 - x = \frac{b}{\frac{1}{2} s}
\end{cases}
\]

\[
\begin{cases}
1 - \frac{a}{\frac{1}{2} s} = \frac{\frac{1}{2} s - a}{\frac{1}{2} s} \\
1 - \frac{b}{\frac{1}{2} s} = \frac{\frac{1}{2} s - b}{\frac{1}{2} s}
\end{cases}
\]

\[
\frac{d}{s} = \frac{\frac{1}{2} s - a}{\frac{1}{2} s} - \frac{\frac{1}{2} s - b}{\frac{1}{2} s}
\]

for both Ratios; therefore

\[
\frac{1}{n} \times \frac{d}{s} + \frac{d^3}{2 s} + \frac{d^4}{3 s^3} + \frac{d^5}{4 s^4} + \frac{d^6}{5 s^5} + \&c. = A = L,
\]

or

\[
\frac{1}{n} \times \frac{d}{s} - \frac{d^3}{2 s} - \frac{d^3}{3 s^3} - \frac{d^4}{4 s^4} - \frac{d^5}{5 s^5} + \&c. = B = L,
\]

\[
\begin{cases}
a & \frac{\frac{1}{2} s}
\end{cases}
\]

\[
\begin{cases}
a & \frac{\frac{1}{2} s} \& b
\end{cases}
\]

**RULE 1.**

\[
\frac{1}{n} \times \frac{2 d}{s} + \frac{2 d^3}{3 s} + \frac{2 d^4}{4 s^3} + \frac{2 d^5}{5 s^5} + \&c. = A + B = L, \text{ Rat. of } a \text{ to } b.
\]

And \( A \times B = L \times \frac{\frac{1}{2} s}{b} = \frac{a}{\frac{1}{2} s} \) or

\[
\frac{1}{n} \times \frac{2 d^2}{2 s} + \frac{2 d^4}{4 s^3} + \frac{2 d^6}{6 s^5} + \&c. = L, \frac{\sqrt[3]{a b}}{\frac{\frac{1}{2} s}} \times \frac{\sqrt[3]{a b}}{\frac{1}{s}}
\]

\[
\frac{d}{s} + \frac{d^3}{4 s^3} + \frac{d^4}{6 s^5} + \frac{d^6}{8 s^7} + \&c. = L, \text{ of the Ratio of the Geometrical to the Arithmetical mean between } (a) \text{ and } (b).
\]

\[\text{But}\]
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But the Difference of the Terms of the Ratio, i.e. \( \frac{1}{4} s^5 \)

\(- a b, \text{ or } \frac{1}{4} a^2 + \frac{1}{2} a b + \frac{1}{4} b^2 = a b = \frac{1}{4} a^2 - \frac{1}{2} a b + \frac{1}{4} b^2 \) (in this Case).

And putting \( \frac{1}{4} s s + a b = y \), \( \text{v} \). (since \( y = s \), and \( d = 1 \)) it follows, that

\[ \frac{\frac{1}{n} \times 2}{y} + \frac{\frac{2}{y^3}}{3y^3} + \frac{\frac{2}{5y^5}}{5y^5} + \frac{\frac{2}{7y^7}}{7y^7} \text{ &c.} = L, \]

of the Ratio between \( \frac{1}{2} s s \) and \( a b \), by Rule 1. Whence

**RULE 2.**

\[ \frac{\frac{1}{n} \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} \text{ &c.}}{y} = L, \]

of the Ratio between \( \frac{1}{2} s \) and \( \sqrt{a b} \). Which Rule is of Excellent use for finding the Logarithms of Prime Numbers, having the Logarithms of the adjoyning Numbers given.

In making Briggs's Logarithms, the Index \( n \) must be 2,3025850, &c. as was hinted before;

For if the Index \( n \) be 100000000 &c. in infinitum,

The Logarithm of 10 will be 2,3025850, &c. as in Naper's;

But that the Logarithm of 10 may be 1000000, &c. as in Briggs's,

The Index \( n \) must be 2,3025850, &c.

And this Index, i.e. Naper's Logarithm of 10, may be easily found several ways, either by the Number 10 it self, or by its Component Parts: But we shall instance only this way.

Since \( 10 = 2 \times 2 \times 2 \times 1 \frac{1}{2} \) \( \Rightarrow 3 \times L \), \( 2 = L, 1 \frac{1}{2} \);

therefore the Log. of 2, and the Log. of \( 1 \frac{1}{2} \) must be found to the Index 1000000, &c.

A a 2   And
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Synopsis Part. 1. Sect. 3,

And Nap. L, 2 is \( \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} \) &c. R. 1,

\[ \therefore 3 L, 2 = \frac{6}{3} + \frac{1}{3} \times \frac{6}{3} + \frac{1}{3} \times \frac{6}{3} + \frac{1}{3} \times \frac{6}{3} \] &c. by Mult.

Also Nap. L, 1 \( \frac{1}{3} \) is \( \frac{2}{9} + \frac{1}{9} \times \frac{2}{9} + \frac{1}{9} \times \frac{2}{9} + \frac{1}{9} \times \frac{2}{9} \) &c. R. 1

But \( \frac{2}{9} = \frac{6}{3} \), \( \frac{2}{9} = \frac{6}{3} \), &c. Therefore.

The Operation stands thus:

<table>
<thead>
<tr>
<th>( \frac{\text{A}}{\text{B}} )</th>
<th>( \frac{\text{C}}{\text{D}} )</th>
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</table>

Therefore Napier's Log. of \( \frac{\text{c}}{\text{c}} \) is \( 2,30256850929940 \).

Or the Index \( n \) for Briggs's Scale of Logarithms, which, by continuing the Operation, is found to be \( 2,3025850929940 \).

\[ \frac{21962962562962}{49382716049} \] &c. Therefore \( \frac{1}{n} \) will be

\[ \frac{86287729760333328}{21962962562962} \] &c.
Example 1.

To find Briggs's Logarithm of 2, only to 10 Places.

Note, That the Index must be assumed of a Figure or two more than the intended Logarithm is to have; therefore, in this Example, \( \frac{1}{\pi} = 0.434294481903 = 2 \), \( d = 1 \), and \( s = 3 \).

\[
\beta = \frac{2}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{5} \times \frac{2}{3} + \frac{1}{7} \times \frac{2}{3} + \text{&c.} = L_2, R_1,
\]

Or
\[
\beta = \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{5} \times \frac{1}{3} + \frac{1}{7} \times \frac{1}{3} + \text{&c.} = \frac{1}{2} L_2.
\]

The Operation stands thus:

<table>
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<tr>
<th>β</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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<th>G</th>
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<th>I</th>
<th>K</th>
<th>L</th>
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</thead>
<tbody>
<tr>
<td>1/3</td>
<td>1,447,648,273,01</td>
<td>1,447,648,273,01</td>
<td>160,849,801,1</td>
<td>160,849,801,1</td>
<td>178,728,009,0</td>
<td>178,728,009,0</td>
<td>198,580,010</td>
<td>198,580,010</td>
<td>220,644,45</td>
<td>220,644,45</td>
<td>245,160,5</td>
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<td>1/3</td>
<td>1,447,648,273,01</td>
<td>1,447,648,273,01</td>
<td>536,166,027,0</td>
<td>536,166,027,0</td>
<td>357,444,018</td>
<td>357,444,018</td>
<td>283,685,72</td>
<td>283,685,72</td>
<td>245,160,5</td>
<td>245,160,5</td>
<td>222,873</td>
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<td>1/3</td>
<td>1,447,648,273,01</td>
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<td>209,53</td>
<td>209,53</td>
<td>201,7</td>
<td>201,7</td>
<td>197</td>
<td>197</td>
<td>19</td>
<td>19</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ \text{Sum} = \frac{1}{2} L_2 = 0.15051497826 \times 2 \]

Therefore, the Logarithm of 2 is \( 0.301029995652 \).

But the same Logarithm may yet be obtained much easier and sooner from this Consideration, viz.

That
Synopsis

Part I. Sect. 3.

That \( \frac{1}{2} \) \( \times \frac{1}{1024} \) \( \times \frac{1000}{1024} \times \frac{1}{1000} = \frac{1}{1024} \).

Therefore \( \frac{L_1 + \frac{1}{2} + \frac{1}{8}}{10} = \frac{L_1 + \frac{1}{10} + \frac{1}{2}}{10} = L_1 \) to 2, \( = L_2 \).

But \( \frac{1000}{1024} = \frac{125}{128} \) \( \times \frac{2}{1} \times \frac{3}{253} \times \frac{3}{5} \times \frac{3}{253} \times \frac{2}{5} \times \frac{3}{253} \).

\( = \frac{125}{128} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{5} \times \frac{3}{5} \times \frac{2}{7} \times \frac{3}{7} \) \( \times \) by \( R_1 \).

Let \( s = q \).

\( \frac{1}{4} \times \frac{1}{2} \times 2 \beta \) \( = A \times 0.01029947387912 \)

\( \frac{1}{3} \times q \times q \times q \times A = B = 48271995 \)

\( \frac{1}{3} \times q \times q \times q \times B = C = 4072 \)

Sum \( L_1 \) to \( = \frac{1}{10} \) \( = 0.010299566398 \).

Add \( L_3 \) \( = \frac{1}{10} \) \( = 3.000000000000000 \).

And \( \frac{1}{3} \) of that \( = 0.03010299566398 \), \( = L_4 \).

Exemplification 2.

To find Briggs's Logarithm of 3.

Here \( a = 2, \) \( s = 4, \) therefore \( \frac{d}{s} = \frac{1}{2} \), conseq. by \( R_1 \).

\( L_3 = \beta \times \frac{2}{1} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{3} + \frac{2}{8} + \frac{1}{32} + \frac{2}{7} \times \frac{1}{128} \).

\( \frac{1}{3} L_3 = \beta \times \frac{1}{3} \times \frac{1}{8} + \frac{1}{5} \times \frac{1}{32} + \frac{1}{7} \times \frac{1}{128} \).

whence the several Terms may be readily found by a Continual Division of \( \frac{\beta}{2} \) by 4, and those again by the Indices of the odd Powers, each respectively, and twice the Sum of the Quotes will be the Logarithm of 3 sought.

But
Chap. 8. Palmariorum Matheseos.

But the Logarithm of 3, is found abundantly sooner, by Rule 2. The adjoining Numbers being 2 (==a) and 4 (==b)

\[ \sum_{\frac{1}{2}} \frac{s}{a} = 3 \]

\[ \sqrt{a \cdot b} = \sqrt{4 \times 2} \]

Aribt. 3

\[ \text{Geom. mean between 2 and 4.} \]

And \( \frac{1}{2} \times 9 = 9 \), and \( \frac{1}{2} \times 3 + a \cdot b = 9 + 8 = 17 = y \).

But \( \frac{\sqrt{4 \times 2}}{\frac{1}{2} s} = \frac{1}{\frac{1}{2} s} \) therefore,

\[ \frac{L_1}{\sqrt{4 \times 2}} + L_2 = \frac{\sqrt{4 \times 2}}{\frac{1}{2} s} = L_2 \]

\[ L_1 = L_2 \]

\[ \frac{1}{\frac{1}{2} s} = \text{L. of 1 to 3, or L. 3.} \]

The L. \( \frac{\sqrt{4 \times 2}}{\frac{1}{2} s} = \frac{1}{17} \times \frac{\beta}{3} + \frac{1}{17} \times \frac{1}{5} \times \frac{\beta}{17} + \frac{1}{7} \times \frac{\beta}{177} \)

And \( L_4 \frac{1}{\sqrt{4 \times 2}} = L_4 + L_2 \)

\[ \frac{1}{2} \times \frac{1}{2} \times \beta = A \]

\[ \frac{1}{2} \times \frac{1}{2} \times A = B \]

\[ \frac{1}{2} \times \frac{1}{2} \times B = C \]

\[ \frac{1}{2} \times \frac{1}{2} \times C = D \]

The Sum is the L. 3

\[ 0.4515449934959 \]

25546734296

294656680

611744

1511

And the same Logarithm may yet be very expeditiously found by means of the Ratio of 2\(^{11}\) to 5 \times 3\(^{8}\) (where d = 37, and s = 65573).

For \( \frac{2^{11}}{5 \times 3^8} \times \frac{5}{2^{11}} = \frac{1}{3^8} \), Therefore,

\[ L_7 \frac{2^{11}}{5 \times 3^8} + L_7 \frac{5}{2^{11}} = L_7 \frac{2^{11}}{5 \times 3^8} + L_7 2^{11} = L_7 \frac{5}{8} \]

But
But \( L_1, 2^{1/5} - L_1, 5 = 3,81647993063 \)

And \( \frac{1}{2} \times 2 \beta \frac{d}{d} = 0,00049010708 \) 1st step of the series.

\[
\text{Sum is } 3,81697003770 = L_1, 3^a
\]

And \( \frac{1}{3} \) thereof is \( 0,4771212471 = L_1, 3^b \).

Here the first step alone gives the Logarithm true to 11 Places.

Such Contractions, to expedite the Operation, may also be found for the other Prime Numbers, by means where-of the Series is made to Converge wonderfully quick: as the Logarithm

\[
\begin{aligned}
\text{Of} & \\
\text{by the ratio of} & \\
7 & \frac{3x^2x5^2}{2x7^2x10^2} \\
11 & \frac{7x11x5^2x2^4}{13x2^5x5^2} \\
13 & \frac{16^3x17^2}{10^3x17^2} \\
17 & \frac{16x2^3x3^4}{17x23x2^4x7^2} \\
19 & \frac{16x17x19x2^4x3^2}{3x7x17x29x2^4} \\
23 & \frac{16x17x19x2^4x3^2}{7x10x11x17x2^2} \\
29 & \quad \text{etc.} \\
31 & \quad \text{etc.} \\
37 & \quad \text{etc.} \\
41 & \quad \text{etc.}
\end{aligned}
\]

**Example 3.**

To find Briggs's Logarithm of the Prime Number 23\(^a\) from the 2d. Rule; as perform'd by Mr. Halley, in the aforementioned Philos. Transi.

The adjoyning Numbers are 22 \((=a)\) and 24 \((=b)\)

\( a = 2, b = 46, \frac{a + b}{2} = 23, \frac{a + b}{2} \) and \( ab = 529 + 528 \)

\( = 1057 = y. \)

But
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But

\[
\frac{1}{\sqrt{s}} \cdot \sqrt{ab} = \frac{\sqrt{24 \times 22}}{\sqrt{s}} + \frac{1}{\sqrt{s}} \cdot \sqrt{24 \times 23} = \frac{L}{\sqrt{s}} = L_{23}.
\]

And \( L, \frac{\sqrt{24 \times 22}}{\sqrt{s}} = \beta + \frac{\beta}{3}y + \frac{\beta}{5}y^2 + \frac{\beta}{7}y^3 + \frac{\beta}{9}y^4 + \ldots \) &c. by \( L, 1 \)

Since \( 2 \times 2 \times 2 \times 3 = 24 \), and \( 2 \times 11 = 22 \), therefore

\[
3L_{2} = L_{3} = L_{24}, \text{ and } L_{2} = L_{11}, = 22. \text{ But } \frac{L_{24}}{2} = L_{22}.
\]

Or \( L, \sqrt{ab} = 1, 36131696126690612945009172659804 \)

\[
\begin{align*}
\frac{1}{\sqrt{s}} \times \frac{1}{\sqrt{s}} = A & = 41087462810146814347215886368 \\
\frac{1}{\sqrt{s}} \times \frac{1}{\sqrt{2}} = B & = 12258521544181829460074 \\
\frac{1}{\sqrt{s}} \times \frac{1}{\sqrt{3}} = C & = 6583235184376175 \\
\frac{1}{\sqrt{s}} \times \frac{1}{\sqrt{4}} = D & = 4208829765 \\
\frac{1}{\sqrt{s}} \times \frac{1}{\sqrt{5}} = E & = 2930
\end{align*}
\]

Sum = \( L_{23} = 1.361727836017592878677711223117 \)

The Ingenious may for his Practice, with the same ease continue the Logarithm to any Number of Places, by taking the Index accordingly; tho' the design of this Treatise, and the narrowness of the Page, determin'd our Example but to few Places; in the Explication of which, we have but the more large, to the end that the Logarithm of any other Incompos'd Number may be made by the foregoing Rules without any farther Direction.

**NOTE.**

1. The Logarithms of Composite Numbers are found by Adding of the Logarithms of their Factors; Thus, the Logarithm of 6 is the Sum of the Logarithms of 2 and 3, for \( 2 \times 3 = 6 \).
2. The Logarithms of the Powers of any Number are obtained by Multiplying the Logarithm of that Number by the Index of the Power; for these Indices are Proportional to those Logarithms: Thus the Logarithms of 4, 8, 16, 32, 64, &c. are found by Multiplying the Logarithm of the Root 2, by the Indices 2, 3, 4, 5, 6, &c. of those Powers respectively. Theref. if \( x^n = a \), then \( n \log x = \log a \).

So that an indifferent Capacity may hence see how the Logarithms of Numbers are made to any exactness or Number of Places; and therefore, how the whole Table of Logarithms may be Calculated anew, or how one already made may be examined, with all desirable Facility and Dispatch.

And as for the Tables themselves; 'tis not thought necessary here to insist on their various Uses, because they are so evident to them that understand what is already said, and so largely handled and exemplified by most Writers of Practical Mathematics.

**Corollary.**

Hence also, from the Logarithm given, 'tis easy to find what Ratio is express'd.

For \( \log (x^n) = n \log x \),

Conf. \( \log (x^n) = n \log (x^n) \),

that is, \( \frac{\log (x^n)}{n} = \log x \),

So that if \( \log (a^n) \) be given, since \( a = 1000 \),

\[ \log (a^n) = n \log a \]

Therefore, if Napier's Log. be given (since \( n = 1000 \) &c.),

\[ \log (a^n) = n \log a \]

Wherefore, if one of the Terms (\( a \) being the Least, and \( b \) the Greatest) of the Ratio, whereof \( L \) is the Logarithm, be given, the other is readily found; for in,

\[ \log (a^n) = n \log a \]

Or
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Or let the next nearest Logarithm be called \( a \), if \( \text{Less, } \beta \) if \( \text{Greater than } L \) (its Num. \( a \), or \( b \)) and \( L - x \), or \( \beta - L = \delta \); then

\[
\frac{a^2}{b^2} \times 1 + \frac{\delta}{2} \cdot \frac{1}{2} \delta^2 + \frac{\delta^3}{3} + \frac{\delta^4}{4} \cdot \frac{1}{3} \delta^3 \cdot L^3, \text{ &c.} = N,
\]

the Number answering to the Logarithm \( L \); where the Series converges according to the smallness of \( \delta \).

And the first step \( a + a \delta, b - b \delta, \) or \( a + n a \delta, b - n b \delta \), serves for the Common Tables.

Also \( a \cdot \frac{a \delta}{1 - \delta} = \frac{1}{1 - \delta} \cdot \frac{a \delta}{1 - \delta} = \frac{1}{1 - \delta} \cdot \frac{a \delta}{1 - \delta} \cdot \frac{1}{1 - \delta} = N. \)

Where the first step only is sufficient for practice, in Tables exceeding any yet extant.

\[\text{i.e. } N = a + \frac{a \delta}{1 - \delta}, \text{ and } b = \frac{b \delta}{1 - \delta} \cdot \]

Or \( \frac{a + \delta}{a} \cdot a + \frac{a - \delta}{a} \cdot b \), and \( \frac{a + \delta}{a} \cdot a + \frac{a - \delta}{a} \cdot b = N. \)

Thereafter, \( \frac{a + \delta}{a} \cdot a + \frac{a - \delta}{a} \cdot b = \left\{ \frac{a}{b} \right\} \cdot N. \)

II. To Extract the Root of an Infinite Equation.

Suppose, for instance, the Infinite Equation to be

\[a x + b x^2 + c x^3 + d x^4 + \ldots, \text{ &c.} = a y + b y^2 + c y^3 + d y^4 + \ldots, \text{ &c.} \]

Put \( x = A y + B y^2 + C y^3 + D y^4 + E y^5 + \ldots, \text{ &c.} \). Then by the first step,

\[a^2 = AA AB, \quad aB = B A^2, \quad \ldots, \quad \frac{a + \delta}{a} \cdot A + \frac{a - \delta}{a} \cdot B = \left\{ \frac{a}{b} \right\} \cdot N. \]

Where it is evident, by comparing the Coefficients, That

\[
\begin{array}{c}
B b 2 \\
A =
\end{array}
\]
\[ A = \frac{a}{a} \]
\[ B = \frac{b - bA^2}{a} \]
\[ C = \frac{y - 2bAB - cA^3}{a} \]
\[ D = \frac{d - bB^3 - 2bAC - 3cA^2B - dA^4}{a} \]
\[ E = \frac{e - 2bBC - 2bAD - 3cAB^3 - 3cA^2C - 4dA^2B - eA^5}{a} \]

&c. Therefore, by Substitution,

\[ q = \frac{a}{a} y \]
\[ + \frac{\beta - bA^2}{a} y^2 \]
\[ + \frac{\gamma - 2bAB - cA^3}{a} y^3 \]
\[ + \frac{\delta - bB^3 - 2bAC - 3cA^2B - dA^4}{a} y^4 \]
\[ + \frac{\epsilon - 2bBC - 2bAD - 3cAB^3 - 3cA^2C - 4dA^2B - eA^5}{a} y^5 \]

&c. Which is the Theorem given for this purpose, by that Ingenious Mathematician Mr. De Moivre, (In Philos. Trans. N° 240.)

And the Observations made for the Continuance of this Series, are hence also manifest; viz.

1. That each Capital Letter is equal to the Coefficient of the Term preceding that where it was first expres'd.

2. That the Denominator of each Coefficient must be always \( a \).
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3. That the 1st. Term of the Numerator must be a respective Coefficients in the Series, \( a y + b y^2, \) &c.

4. That the Capital Letters must be Combin'd, as often as the Sum of their Exponents can be made equal to the Index of the Power to which they belong.

5. That there must be so many Capitols in each Term, as are denoted by the Exponents of the small Letter annex'd.

6. That the Capitols of every Member are capable of so many Permutations as are express'd by the Numeral Figures prefix'd.

**NOTE 1.** The Quantities \( a, b, c, \) &c. \( a, b, y, \) &c. are taken at Pleasure; Therefore may each represent an Infinite Series (if need be), or any indetermined Quantities: Consequently, if the Equation involve more than two indetermined Quantities, such also may easily be deduced from hence.

**NOTE 2.** This Theorem may be made infinitely more General, by Substituting \( y^n, y^{2n}, y^{3n}, \) &c. in the room of \( y, y^2, y^3, \) &c. and putting general Exponents for particular ones in the other Indeterminates, then proceed as these Exponents require.

And those also may be made infinitely more General, by observing the foregoing Method.

III. To Investigate General Ways of Extracting the Roots of all sorts of Equations.

Suppose any Equation whatsoever, as,

\[ x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + \cdots d x^{n-4} + \cdots + A = 0 \]

Whence \( A \) is the Absolute known Quantity;

\( x \) the Root required;

\( n \) the Index of the Highest Power;

\( 1, a, b, c, \) &c. the respective Coefficients.
Put \( k \) (a known quantity, taken at Pleasure, tho’ the nearer the True Root the better) \( \pm u \) (an unknown quantity) equal to \( x \); then the Equation will be

\[ \pm 1 \times k^{n-1} - \frac{n}{1} x \times \frac{n-1}{2} k^{n-2} x u + \text{&c.} \]

\[ \pm a \times k^{n-1} - \frac{n}{1} x \times \frac{n-1}{2} k^{n-2} x u + \text{&c.} \]

\[ \pm b \times k^{n-1} - \frac{n}{1} x \times \frac{n-1}{2} k^{n-2} x u + \text{&c.} \]

\[ \pm c \times k^{n-1} - \frac{n}{1} x \times \frac{n-1}{2} k^{n-2} x u + \text{&c.} \]

\[ \pm \text{&c.} \]

\[ \pm A = \text{&c.} \]

1. Now by rejecting the Powers of \( u \), and their Coefficients, it may be expressed by a Simple Equation, and its True value found by repeating the Operation so far as is necessary.

For since \( u \) is greater than it should be when it has a Positive, but less when it has a Negative sign, therefore ’tis plain, that if the supposed Root be less than the required Root, the following Operation will make it greater, and if it be greater, ’twill continue so, tho’ less than the former, therefore the Root must necessarily Converge; and after an Infinite Convergence (if need be) must become equal to that sought: And therefore also the rest of the of the Terms may be safely rejected.
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Theret, \( u = \frac{A - k^n + a k^{n-1} - b k^{n-2} + c k^{n-3} + \cdots}{n k^{n-1} + n - 1 a k^{n-2} + n - 2 b k^{n-3} + n - 3 c k^{n-4} + \cdots} \), &c.

And since \( k + u = x \), by Supposition, therefore

\( x = \frac{A + n - 1 k^n + n - 2 a k^{n-1} + n - 3 b k^{n-2} + n - 4 c k^{n-3} + \cdots}{n k^{n-1} + n - 1 a k^{n-2} + n - 2 b k^{n-3} + n - 3 c k^{n-4} + \cdots} \), &c.

Whence Particular Theorems, (agreeing with those gi- by the Learned and Ingenious Mr. Reaphis, in his Analysis Equationum) are easily drawn, for Extracting the Roots of all sorts of Equations, however Compounded or Affected.

And whatever Term is wanting in the Equation, must be omitted in the Theorem.

As 1. For all Pure Powers, i.e. if \( x^n = A \),

Then \( u = \frac{A - k^n}{n k^{n-1}} \); and \( x = \frac{A + n - 1 k^n}{n k^{n-1}} \).

2. If \( x + a x = A \), then,

\( u = \frac{A - k^2 + a k}{2 k + a} \); and \( x = \frac{A + k^2}{2 k + a} \).

And the like in any other Equation.

**NOTE.** That \( k \) may be taken at Pleasure; then in renewing the Operation, if you use the

**First Theorem, Let**

\[
\begin{align*}
1. f. \quad k & = 1. f. \\
2. d. \quad u & = 2. d. \\
3. d. \quad x & = 3. d.
\end{align*}
\]

**Second Theorem, Let**

\[
\begin{align*}
1. f. \quad k & = 1. f. \\
2. d. \quad x & = 2. d. \\
3. d. \quad x & = 3. d.
\end{align*}
\]

II. Since \( x = k + u \), by Supposition; Therefore

\[
x^n = k^n \cdot \frac{n}{1} k^{n-1} u + \frac{n}{1} k^{n-2} u u + \cdots;
\]

And
And if \( x^n \) be greater than \( k^s \), 'twill be \( k^n + u \):

Let \( (x^n + k^n) = \frac{n}{1} k^{n-1} u + \frac{n-1}{2} k^{n-2} u \), etc. = \( m \)

Therefore \( m \) taken equal to

\[
\begin{align*}
&\frac{n}{1} k^{n-1} u \quad \text{(Doubles)} \\
&\frac{n}{1} \times \frac{n-1}{2} k^{n-2} u \quad \text{(Triples)} \\
&\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} k^{n-3} u \quad \text{(Quadr.)} \\
&\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} k^{n-4} u \quad \text{(Quint.)} \\
&\text{etc.} \\
\end{align*}
\]

the True Figures in the assumed \( k \), at each Operation.

Since, \( u = \frac{m}{nk^{n-s}} \), or \( \frac{m}{nk^{n-1} + \frac{1}{2} nn - nk^{n-s} u} \)

Therefore 'tis equal to

\[
\frac{m}{nk^{n-1} + \frac{1}{2} n^2 - nk^{n-2} x + \frac{m}{nk^{n-1}}} \quad \text{or} \quad \frac{m}{nk^{n-1} + \frac{n-1}{2 k}}
\]

That is \( u = \frac{mk}{nk^{n-1} + \frac{1}{2} n - 1 m} \), which is the Rational

Theorem given by Mr. Halley, (in Philos. Trans. N. 210.)

for Extracting the Roots of all Pure Powers.

Also
Also, \[ \pm n^{k-1}u + \frac{1}{n}n^{n-n}k^{n-2}u = \frac{m}{2} \text{ by Hyp.} \]

Then \[ u = \frac{2k}{n-1} - \frac{m}{\frac{1}{2}n-n^k-2} \]

by Equal Division.

And \[ u = \frac{k}{n-1} = \frac{k\frac{1}{n} - \frac{m}{\frac{1}{2}n-n^k-2}}{2} \]

Therefore,

\[ (k^a + m) \text{ or } k^a u = \frac{n-2}{k} + \frac{1}{n-1}k^a + \frac{m}{\frac{1}{2}nn-n^k-2} \]

Which is the Irrational Theorem given for the same purpose.

Note: If the Root of a very high Power be required; the renewing the Theorem will be somewhat troublesome.

Therefore having found 3 or 4 Figures of \( u \), the rest may be attained much easier, by applying the following Correction, viz. if it be \( k^a u \).

\[ \frac{1}{n-1}k^a + \frac{1}{\frac{1}{2}nn-n^k-2} \]

In repeating the Correction, the last found must be used; and in the 1st, 2nd, 3rd &c. Correction, the DAP must have 3, 4, 6, &c. Terms.

Also, the Divisor must be corrected every operation, by Subduing, or Adding the last Correction, from, or so it, if it be \( k^a u \).

Thus, if \( x^{14} \) is found, with wonderful facility, by only one Supposition of \( x = 1 \), and Corrections, that \( x = 1 \), \( \ldots \), which would have bid an intolerable Labour to perform by any other Method.

\( \therefore \) III. Whence
III. Whence also may be had variety of other Theorems for finding the Roots of Equations: By resuming the former General Equation, where

\[ A = p \cdot q \cdot u \cdot r \cdot u^2 \cdot s \cdot u^3 \cdot t \cdot u^4 \cdot \text{&c.} \]

And for Equations under the 5th Dimension, the following Table will abundantly assist the Practitioner; where

\[ x^9 \cdot a \cdot x^8 \cdot b \cdot x^7 \cdot c \cdot x^6 \cdot d \cdot x^5 \cdot e \cdot x^4 \cdot f \cdot x^3 \cdot g \cdot x^2 \cdot h \cdot x \cdot A = 0. \]

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<td>8 a k^7</td>
<td>7 b k^6</td>
<td>6 c k^5</td>
<td>5 d k^4</td>
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<tr>
<td>r</td>
<td>36 k^7</td>
<td>28 a k^6</td>
<td>21 b k^5</td>
<td>15 c k^4</td>
<td>10 d k^3</td>
<td>6 e k^2</td>
<td>3 f k^1</td>
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<td>s</td>
<td>84 k^6</td>
<td>70 a k^5</td>
<td>56 b k^4</td>
<td>42 c k^3</td>
<td>30 d k^2</td>
<td>20 e k^1</td>
<td>15 f k^0</td>
</tr>
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<td>t</td>
<td>126 k^5</td>
<td>105 a k^4</td>
<td>90 b k^3</td>
<td>70 c k^2</td>
<td>56 d k^1</td>
<td>42 e k^0</td>
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<td>u</td>
<td>126 k^4</td>
<td>105 a k^3</td>
<td>90 b k^2</td>
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<td>w</td>
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<td>y</td>
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</tr>
<tr>
<td>t</td>
<td>9 k</td>
<td>a</td>
<td>x u^8</td>
<td></td>
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</tr>
</tbody>
</table>

1. Observe, That the Terms in the Ranks \( p, q, r, \) &c. must have the same Signs with their respective ones in the Equation.

2. If \( b \leq k \leq p \), then \( k \div u = x \).

3. If \( b = k \leq u \), then the Terms in the Ranks \( q, s, v \), must have the same Signs with, or different from their respective ones in the given Equation.

And since \( p \cdot q \cdot u \cdot r \cdot u^2 \cdot s \cdot u^3 \cdot t \cdot u^4 \), &c. \( A = 0 \). Then \( p = q \cdot u \cdot r \cdot u^2 \), &c. Therefore \( u \) taken to

\[
\begin{align*}
&\left\{ \begin{array}{l}
\text{Doubles} \\
\text{Triples} \\
\text{Quadr.} \\
\text{Quins.} \\
\&c.
\end{array} \right.

\begin{align*}
&\text{The True Figures in} \\
&\text{the assumed } k, \text{ at each Operation.}
\end{align*}
\]

Since
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Since \( u = \frac{p}{q} \) or \( \frac{p}{q \pm ru} \) or \( \frac{p}{q \pm rp} = \frac{qp}{qq \pm rp} \)

Thereof, \( x = k \pm \frac{qp}{qq \pm rp} \); which is the Rational Theorem for finding the Roots of Affected Equations of what Power soever.

Also because \( q u \pm r u^2 = \pm p \) by Supposition.

Therefore, if \( p \) and \( A \) have like, or unlike; or \( p \) and \( r \), unlike, or like Signs, 'twill be \( k + u \), or \( k - u \); and

\[
u = \frac{\frac{1}{2} q \pm \frac{q q \pm p r}{r}}{\frac{1}{2}}; \quad x = k \pm \frac{\sqrt{\frac{1}{2} q q \pm p r}}{r}\]

which is the Irrational Theorem for solving Affected Equations.

And by renewing the Calculation (if need be) and taking always the \( x \) last found for the \( k \) in the New Equation, 'tis easy to proceed to any exactness. But instead of renewing the Theorem, you may advantageously use the following Correction.

\[
s u^3, t u^4, v u^5, w u^6, \ldots, \frac{1}{2} s u^3, \frac{1}{2} t u^4, \frac{1}{2} v u^5, \ldots\]
\[
viz. \quad 2 \times \frac{\sqrt{\frac{1}{2} q q \pm p r}}{r}\]

according as \( q \) and \( r \) have like, or unlike Signs. Where, in the Dividend, those Terms of the Equation that have the same Signs with, or different from \( s \), must be \( + \), or \( - \).

And, in the Divisor, \( \frac{1}{2} q q \pm r \) contrary to, or the same with the Sign \( s \), if it be \( k - u \), but on the contrary, if it be \( k + u \).

Also, in repeating the Correction, the Divisor may be corrected by Subtracting, or Adding the last Correction multiplied by \( r \), from, or to \( u \), if \( s \) be Affirmative, or Negative;
Synopsis Part I Sect 3.

Eavitve; Then if q and r have the same, or different signs, the new Correction must have the same signs with, or different signs from s.

\[ F \times A M P L E. \]

Suppose the value of x in this Cubic Equation
\[ x^3 + 438 x^2 - 7825 x - 98508430 = 0. \]
or \( x x x + a x x - b x - A = 0. \) was required.

Let \( x = k \pm u; \) (which \( k \) should be assum'd as near as possible to \( x \), because convenient, but not necessary);

1. Suppose \( k = 300; \) then
\[
\begin{align*}
x^3 & = + 27000000 - 270000 u + 900 u^2 + u^3, \\
a x^2 & = + 39420000 + 262800 u + 438 u^2, \\
b x & = - 2347500 - 7825 u, \\
A & = - 98508430 \\
\text{i.e.} & \frac{3435930}{p} + \frac{5249750}{q} + \frac{1338 u^3}{ru} = 0. \end{align*}
\]

Then by the Rational Theorem
\[
\frac{u}{p} \left( \frac{p^2}{q} + \frac{r p}{q} \right) = 56.2 \text{; theref. } x = k + u = 356.2.
\]

And by the Irrational Theorem
\[
\frac{u}{p} \left( \frac{9 q + r p}{q} - q \right) = 57. \text{; theref. } x = k + u = 357.5.
\]

2. Renew the Operation; And let \( k = 356; \) then,
\[
\begin{align*}
x^3 & = + 54518016 + 380208 u + 1c68 u^2 + u^3, \\
a x^2 & = + 55510368 + 311856 u + 438 u^2, \\
b x & = - 2785700 - 7825 u, \\
A & = - 98508430 \\
\text{i.e.} & \frac{665746}{p} + \frac{684239}{q} + \frac{1506 u^3}{ru} = 0. \end{align*}
\]

Then
Then by the Rational Theorem

\[ u \left( \frac{p}{q} - \frac{rp}{q} \right) = 970894 \quad \therefore x = 356, 970894 \]

And by the Irrational Theorem

\[ u \left( \frac{\frac{1}{2} q^2 - \frac{1}{2} q r p}{r} \right) = 970898 \quad \therefore x = 356, 970898 \]

Or supposing \( k = 357 \), Then,

\[
\begin{align*}
\frac{x^3}{p} & = \frac{45492993}{292347} - 382347 u + 1071 uu \\
\frac{ax^2}{q} & = \frac{55822662}{2793528} - 312732 u + 438 uu \\
-bx & = \frac{2793528}{7825u} \\
-A & = \frac{-98508430}{} \\
\end{align*}
\]

i.e. \[ \left\{ \begin{array}{l}
\frac{1}{p} - \frac{qu}{q} + \frac{ru}{u} = 0 \\
\therefore x = k - u
\end{array} \right\} \]

Then by the Rational Theorem

\[ u \left( \frac{p}{q} \right) = 02910318169 \quad \therefore x = 356, 9708968183 \]

And by the Irrational Theorem

\[ u \left( \frac{\frac{1}{2} q - \frac{1}{2} q r p}{r} \right) = 02910318180, \text{ therefore,} \]

\[ x = 356, 9708968182 \]

If more Accuracy were required, repeating the Operation, or applying the Correction, would give any desired Number of Figures true in the Root.

The vast Advantage of this Method, beyond any other whatever, will be evident to them that judiciously examine each; who will also find it, considering the several Compendiously easy ways of Operation, together with
CHAP. IX.

OF GEOMETRIC PROGRESSION.

DEFINITION.

A Continued Geometric Proportion, that is, where the Terms do Increase or Decrease by Equal Ratio's, is called a Geometric Progression; thus,

\[ a, ar, arr, arrr, \text{&c. Incr. from a Continual Mult.} \]
\[ a, \frac{a}{r}, \frac{a}{rr}, \frac{a}{rrr}, \text{&c. Decr. Divis.} \]

SCHOLIUM 1.

But since this Progression is only a Compound of two Series, viz.

Of \[ \text{Geom. Proport. } 1, r, r^2, r^3, r^4, r^5, \text{&c.} \]

Therefore, the most Natural Progression is that which begins with 1.

As \[ \frac{1}{1}, \frac{r}{1}, \frac{r^2}{1}, \frac{r^3}{1}, \frac{r^4}{1}, \frac{r^5}{1}, \text{&c. Increasing.} \]

i.e. \[ 1, r, r^2, r^3, r^4, r^5, \text{&c.} \]

And \[ \frac{1}{1}, \frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \frac{1}{r^4}, \frac{1}{r^5}, \text{&c. Decreasing.} \]

i.e. \[ 1, \frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}, \frac{1}{r^4}, \frac{1}{r^5}, \text{&c.} \]
In Geometric Progression;

\[
\begin{align*}
\begin{cases}
a \\ r \\ n \\ l \\ s
\end{cases}
& \text{be the} \\
& \text{First Term.} \\
& \text{Ratio.} \\
& \text{Number of Terms.} \\
& \text{Last Term.} \\
& \text{Sum of all the Terms.}
\end{align*}
\]

Then any three of these Terms being given, the other two are easily found.

And the several Cases are reducible to Ten Propositions, which are all Solved by the Two following Lemmata.

I. Of Increasing Progressions.

**Lemma 1.**

In an Increasing Geometric Progression.

\[
a, ar, ar^2, ar^3, ar^4, ar^5, \text{ &c.}
\]

'Tis \(1 : r : : s - l : s - a\).

\[
\begin{align*}
\text{For } a & : ar :: 1 - l : s - a, \\
\text{But } a & : ar :: 1 : r,
\end{align*}
\]

\[
\text{Therefor } 1 : r :: s - l : s - a.
\]

**Corollaries.**

1. \(s = \frac{rl - a}{r - 1} = \frac{l - a}{r - 1} + l\).

2. \(r = \frac{s - a}{s - l} = s - a \times \frac{1}{s - l}\).

3. \(a = s + rl - rs = rl - s \times r - 1\).

4. \(l = \frac{rs - s + a}{r} = \frac{a + r - 1 \times s}{r} = s - \frac{s - a}{r}\)

**Lemma.**
SYNOPSIS

Part I. Sect. 2.

LEmma 2.

In an increasing geometric progression.

Tis \( l : r^{n-1} : : a : l \).

For \( a, ar, ar^2, \ldots, ar^{n-1} \).

Therefor \( l : r^{n-1} : : a : l \).

C R O L L A R I E S.

1. \( l = ar^{n-1} = a \times r^{n-1} \)
2. \( a = \frac{l}{r^{n-1}} = l \times \frac{1}{r^{n-1}} \)
3. \( n = \frac{L_2 r^{n-1}}{L_2 r} + 1 = \frac{L_2 l - L_2 a}{L_2 r} + 1 \)
4. \( r = \frac{l}{a} \times \frac{1}{r^{n-1}} \)

P R O P O S I T I O N S.

I. Given \( a, r, n \); Required \( l, s \).

1. \( l = ar^{n-1} (= a \times r^{n-1}) \) by Lem. 2.

But \( s = \frac{r l - a}{r - 1} \) by Lem. 1. And \( r \times l = ar^n \) by Mult.

2. Therefor \( s = \frac{ar^n - a}{r - 1} (= a \times \frac{r^n - 1}{r - 1}) \) by Substitution.

II. Given \( a, r, l \); Required \( s, n \).

1. \( s = \frac{rl - a}{r - 1} (= \frac{l - a}{r - 1} + l) \) by Lem. 1.

2. \( n = \frac{L_2 l - L_2 a}{L_2 r} + 1 \) by Lem. 2.
III. Given \( a, r, s \); Required \( l, n \).

1. \[ r - \frac{1}{r} \left( \frac{a}{s} \right) \] (\( = ar^{-1} \)) by Lem. 1st. and 2d.

Then \[ r - \frac{1}{r} \left( \frac{a}{s} \right) \] (\( = r \times ar^{-1} \)) \( = ar^n \) by Multi.

And \[ r^n = \frac{r - 1 \times s + a}{a} \] by Divis. But \( n = \frac{r - 1 \times s + a}{a} \).

2. Theref. \( l = \frac{r - 1 \times s + a - L_s}{r} \) by Division.

IV. Given \( a, l, s \); Required \( r, n \).

1. \[ r = \frac{s - a}{s - l} \left( = s - a \times \frac{1}{s - l} \right) \] by Lem. 1.

2. And \( n = \frac{L_r - 1 \times a}{L_r} \) (by L. 2)

V. Given \( a, n, s \); Required \( r, l \).

Since \[ \frac{sr - s + a}{r} \] (\( = l \)) \( = ar^{-1} \) by Lem. 1st. and 2d.

Then \( sr - ar^n = s - a \) by Divis. and Transp.

1. Theref. \( -r^n + \frac{s - a}{a} = \frac{s - a}{a} = \frac{s}{a} - 1 \), by Divis.

And since \( l = \frac{s - a}{s - l} \), theref. \( l = \frac{s - a}{s - l} \).

2. Theref. \( l \times s - l \) \( = a \times s - a \) by Multi.
VI. Given $a, n, l$, Required $r, s$.

1. $r = \frac{l}{1 - a}^{\frac{-1}{l-1}}$ by Lem. 2. But $\frac{l}{r-1} = s$ by Lem. 1.

2. Thereof, $s = \frac{l - a}{l - a^{\frac{1}{n-1}}} + l$, by Substitution.

VII. Given $r, n, l$; Required $a, s$.

1. $a = \frac{l}{r - 1}$ by Lem. 2. But $\frac{lr - a}{r - 1} = s$, by Lem. 1.

2. Thereof, $s = \frac{lr - l}{r - 1} = \frac{lr^n - l}{r^n - r^{n-1}}$ by Substitution.

VIII. Given $r, n, s$; Required $a, l$.

Since $sr - s + a = ar^n$ by Lem. 1st. and 2d.

Then $sr - s = ar^n - a = a(r^n - 1)$ by Transp.

1. Thereof, $a = \frac{sr - s}{r^n - 1} = \frac{r - 1}{r^n - 1} \times s$ by Division.

And since $s = \frac{lr^n - 1}{r^n - r^{n-1}}$ by Prop. 7. of $sr - s = r^n - l$.

2. Thereof, $l = \frac{sr^n - sr^{n-1}}{r^n - 1}$ by Division.

IX. Given $r, l, s$; Required $a, n$.

1. $a = s + rl - rs (= lr - s \times r - 1)$ by Lem. 1.

But $\frac{l}{a} (= s + rl - rs) = r^n - 1$ by Lem. 2.
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And \( L \frac{1}{s + r l - r s} = \frac{y - 1}{L - r} \) by the Nat. of Log.

2. \( \text{Theref. } n = \frac{L - L}{L - s + r l - r s} = \frac{1}{L - r} \) by Div. and Tr.

X. Given \( n, l, s \); Required \( r, a \).

Since \( sr^n - sr^{n-1} = lr^n - l \) by Prop. 8.

Then \( s - r^{n-1} - sr^n + lr^n \) by Trans.

1. \( \text{Theref. } r^n + \frac{s}{s - r^{n-1}} \left( s - \frac{1}{s - l} \right) = \frac{1}{s - l} \)

2. \( a \times s - a^{n-1} = l \times s - l^{n-1} \) by Prop. 5.

II. Of Decreasing Geometric Progressions.

In \textit{Finite Decreasing Progressions}, the same Rules will serve for the like Propositions, if the Series be inverted, so that the \textit{Least Term} be the \textit{First}, and the \textit{Greatest} the \textit{Last}.

And since in the \textit{Increasing Geometric Progression}

\[ a, ar, ar^2, ar^3, ar^4, ar^5, \text{ &c. to } ar^{n-1} = l. \]

\( \text{Tis } r - 1 : 1 \cdots l - a : s - l. \)

Therefore in a \textit{Decreasing Geometric Progression}

\( \text{Tis } r - 1 : 1 \cdots a - l \cdots s - a. \) by Inverting the Terms

\textbf{C O R O L L A R I E S.}

1. But in an \textit{Infinite Decreasing Progression} \( l = 0; \)

Therefore \( r - 1 : 1 \cdots a : s - a. \) whence,

\( \textit{Dd 2} \)

\textit{Given}
2. Also \( a = \frac{a}{r} \) (i.e. \( 1st. = 2d. : 2d. or x :: r - 1 : 1 \)).

Thereof \( a = x : x :: \frac{a-1}{r} : s \), in \( \frac{a \text{ Finite} \text{ Progr.}}{a \text{ Finite Progr.}} \)

And \( s = \frac{a}{a-x} \) in a Finite, or \( s = \frac{aa}{a-x} \) in an Infinite Decreasing Progression.

**Question.** Suppose a Body should move at this rate, viz. in the 1st. Moment 10 Miles, in the 2d. 9 Miles, in the 3d.

\( \frac{1}{2} \) Sc. eternally, as 10 to 9;

Here is given \( r = \frac{10}{9} \), \( a = 10 \); required \( s \); then,

By Cor. \( \sum_{1}^{1} s = \sum_{2}^{a} \frac{ra}{a-x} = \frac{100 \text{ Miles sought.}}{100} \)

That is, a Moveable Body continuing its Motion in that Ratio eternally, would only run 100 Miles, or more than any thing that is less than 100 Miles.

3. Since \( r - 1 : 1 :: a : s - a \), therefore,

\( s - a = \frac{a}{r - 1} = \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} \), &c. \( \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} \) &c.

Whence, if any Quantity \( a \) be continually divided by any other Quantity \( r \), the Sum of all the Terms will be \( \frac{a}{r - 1} \)

\( \text{i.e.} \)
i.e. \( \frac{1}{r-1} \times a \) or the \( \frac{1}{r-1} \) of \( a \).

Therefore, \( a \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} \), \&c. = s \times \frac{1}{r-1}

Or, \( \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} \), \&c. = \frac{1}{r-1}

Where, if \( a = r - 1 \), then \( s = 1 \).

4. Whence, 'tis evident, That an Infinite Progression, or an Infinitely Infinite one, may be Collected into one Sum; which Sum may not be only Finite, but equal to Nothing.

And of Infinites 'tis hence plain, that some are equal, others unequal; and also that one Infinite may be equal to Two or more Finites, or Infinites.

---

**CHAP. X.**

**Of Interest.**

I. *Of a Single Sum of Money Paid either Before, or After 'tis Due.*

1. **Allowing Simple Interest:**

Put \( p = \text{Principal, or Sum forborne}. \)

\( n = \text{Number of Years, or parts of a Year}. \)

\( r = \text{Rate of } \frac{1}{1}, \text{per Annum.} \)

\( m = \text{Amount of the said Principal, for that Time, at that Rate}. \)

Since
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Since the Amount of 1 l. for 1 Year, is \(1 + r\);
Therefore the Amount of 1 l. for \(n\) Years is \(1 + nr\);
And \(1_l = 1 + nr \propto p = p + pnr = m\), the Amount,
and \(pnr = \text{Interest of the Principal } p\), at the Rate \(r\), in \(n\) Years. Hence,

<table>
<thead>
<tr>
<th>Probl.</th>
<th>Given</th>
<th>Regd.</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(p, r, n)</td>
<td>(m)</td>
<td>(p \times (1 + rn))</td>
</tr>
<tr>
<td>2.</td>
<td>(m, r, n)</td>
<td>(p)</td>
<td>(m ÷ (1 + rn))</td>
</tr>
<tr>
<td>3.</td>
<td>(m, p, r)</td>
<td>(n)</td>
<td>(m - p ÷ pr)</td>
</tr>
<tr>
<td>4.</td>
<td>(m, p, n)</td>
<td>(r)</td>
<td>(m - p ÷ pn)</td>
</tr>
</tbody>
</table>

Whence variety of easy Rules for Practice may be deduced; such as, To find the Interest of any Sum, at any Rate, for any Number of days.

\(\frac{365 \times 100}{7300}\)

'Tis plain, that \(n\) Days \(\times p\) Pence \(÷ 7300\) is the Interest in Pence, at 5 l. per Cent. per An. Therefore,

\(\frac{2r \times \frac{np}{7300}, or 2r \times \frac{np}{73}}{or 2r \times \frac{np}{7300} ÷ 10.}\) is the Interest in Pence, at the Rate \(r\).

2. Allowing Compound Interest.

Put \(x (= 1 + r) = \text{Principal and Interest of 1 l. for any given Time, at any given Rate}.\) Then,

Since, \(1_l: x: : p: p \times = 1\text{st. Year Amount} (m).\)
\(1_l: x: : p \times : p \times^2 = 2\text{nd. Year Amount} (m).\)
\(1_l: x: : p \times^2: p \times^3 = 3\text{rd. Year Amount} (m).\) &c. &c.

Therefore 'tis \(p \times^r = m\); whence, Prob.
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<table>
<thead>
<tr>
<th>Probl.</th>
<th>Given</th>
<th>Req'd.</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(p, n, x)</td>
<td>(m)</td>
<td>(p \times n^x) or (L, p + \frac{1}{L}, x \times n)</td>
</tr>
<tr>
<td>2.</td>
<td>(m, n, x)</td>
<td>(p)</td>
<td>(m \div x^n) or (L, m - \frac{1}{L}, x \times n)</td>
</tr>
<tr>
<td>3.</td>
<td>(p, m, x)</td>
<td>(n)</td>
<td>(x^\frac{m}{n} \div x) or (L, x^\frac{m}{n} L, x)</td>
</tr>
<tr>
<td>4.</td>
<td>(p, m, n)</td>
<td>(x)</td>
<td>(m^\frac{p}{n}) or (L, m - \frac{1}{L}, p \div n)</td>
</tr>
</tbody>
</table>

Quest. 1. Suppose the Principal \(p\) were at Interest for \(n\) Times at the Rate \(x\); what is the Amount \(m\)?

Quest. 2. If the Sum \(m\) was to be paid \(n\) Times hence; what is its present Value \(p\) discounting at the Rate \(x\)?

Quest. 3. If the Principal \(p\) at the Rate \(x\) gives the Amount \(m\); what is the Time of forbearance \(n\)?

Quest. 4. If the Principal \(p\) for \(n\) Times gives the Amount \(m\); what is the Rate \(x\) of Interest?

II. Of several Equal Payments at several Equal Times.

First, When paid after they are due;

Or Rules for finding the Amount, &c. of Annuities, Pensions, or Rents, &c. in Arrear.

1. Allowing Simple Interest.

Put \(a\) for the Annuity, Rent, or Pension; then,

Tis evident, \(1l. : r :: aL : ar\), that is,

The Interest of \(aL\) at the Rate \(r\), per \(1l.\) per \(An.\) is \(ar\).

But \(\left\{ \frac{a + 1}{a + 2} \right\} \frac{ar}{x} \) is the \(\text{1st.}\) \(\text{2d.}\) \(\text{3d.}\) \(\text{nth.}\) Years Amount (m).

Therefore
Therefore \( na + \frac{nn-n}{2} ar \) is the Sum of those Amounts, or the Amounts (m) of the Annuity (a) at the \( n \)th Year's end; whence,

<table>
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<th>Requd.</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( a, n, r; )</td>
<td>( m )</td>
<td>( na + \frac{nn-n}{2} ar )</td>
</tr>
<tr>
<td>2.</td>
<td>( m, n, r; )</td>
<td>( a )</td>
<td>( \frac{2m}{2+nr-rx n} )</td>
</tr>
<tr>
<td>3.</td>
<td>( m, n, a; )</td>
<td>( r )</td>
<td>( \frac{m - na x 2}{n - 1 \times n a} )</td>
</tr>
<tr>
<td>4.</td>
<td>( m, r, a; )</td>
<td>( n )</td>
<td>( \frac{r + r + 8mr a}{2 ra} )</td>
</tr>
</tbody>
</table>

Let \( 2a - ra = ? \)

2. Allowing Compound Interest.

Since the Last Year's Annuity (a) tarries out no Time, therefore no Interest can be demanded for it; consequently the First Year's Annuity will become \( ax^{n-1} \).

Therefore, \( a + ax + ax^2 + ax^3 + ax^4 \), &c. to \( ax^{n-1} \) = m the Amount.

But \( a : ax^2 : : m - ax^{n-1} : m - a \), by Cor. 13. Th. 2.

That is \( 1 : x \frac{3}{3} : : m - ax^{n-1} : m - a \), by Cor. 13. Th. 2.

Therefore \( mx - ax^n = m - a \), whence,
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<table>
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<th>Reqd.</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>a, x, n</td>
<td>m</td>
<td>$\frac{x^n - 1 \times a}{x - 1}$</td>
</tr>
<tr>
<td>2.</td>
<td>m, x, n</td>
<td>a</td>
<td>$\frac{x - 1 \times m}{x^n - 1}$</td>
</tr>
<tr>
<td>3.</td>
<td>m, x, a</td>
<td>n</td>
<td>$\frac{L \times (x - 1) \times m + a - L \times a}{L \times x}$</td>
</tr>
<tr>
<td>4.</td>
<td>m, a, n</td>
<td>x</td>
<td>$-x^n + \frac{m}{a} x = \frac{m - a}{a}$</td>
</tr>
</tbody>
</table>

**Quest. 1.** If the Annuity a be forborn n Times, at the rate r; what Amount m will it arise to?

**Quest. 2.** If in n Times forbearance, at the Rate r the Amount m is raised; what was the Annuity a forborn?

**Quest. 3.** If the Annuity a, at the Rate r, raise or amount to m; what was the Time n of forbearance?

**Quest. 4.** If the Annuity a forborn n Times, raise the Stock or Amount m; what was the Rate r of Interest?

Secondly, **Being paid before they are due.**

Or Rules for finding the Discount, &c. in Buying and Selling of Annuities, Pensions and Leases in Reversions, &c.

**1. Allowing SIMPLE INTEREST.**

Since, As the Amount of £1 for any Time is to 11; So is the Amount of an Annuity, to its Present value.

\[ i, \text{c. } 1 + nr : 1 :: \text{na} + \frac{nn - n}{2} ar : \frac{nn - nx ar}{1 + nr} = c, \]

\[ \text{E.c.} \]

Therefore
Therefore,

<table>
<thead>
<tr>
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<th>Given</th>
<th>Reqd.</th>
<th>Solution</th>
</tr>
</thead>
</table>
| 1.     | $a, r, n;$  | $s.$  | \[
\frac{2n + nr - nr \times a}{2 + nr}
\] |
| 2.     | $s, r, n;$  | $a.$  | \[
\frac{2 + nr \times s}{2 + nr - r \times n}
\] |
| 3.     | $s, n, a;$  | $r.$  | \[
\frac{na - s \times 2}{2s - an + a \times n}
\] |
| 4.     | $s, r, a;$  | $n.$  | \[
\frac{7 + 3r + 8sar^{\frac{1}{2}}}{2ra}
\] |

Let $2sr - ra - 2a = ?$

2. **Allowing Compound Interest.**

\[
\begin{align*}
x \cdot 1 \cdot 1 : a : \frac{a}{x} \\
x \cdot 1 \cdot 1 : \frac{a}{x} : \frac{a}{x^2} \\
x \cdot 1 \cdot 1 : \frac{a}{x^2} : \frac{a}{x^3}
\end{align*}
\]

Since,

\[
\begin{align*}
x : 1 \cdot 1 : \frac{a}{x} & = \text{Present Value (s) at the} \\
x : 1 \cdot 1 : \frac{a}{x^2} & = \text{Year's end.} \\
x : 1 \cdot 1 : \frac{a}{x^3} & = \text{&c.}
\end{align*}
\]

Thereof, \[
\frac{a}{x} + \frac{a}{x^2} + \frac{a}{x^3} + \&c. = \text{Present value of } s.
\]

But \[
\frac{a}{x} : \frac{a}{x^2} : \frac{a}{x^3} = s - \frac{a}{x} \text{ by Cor. 13. Tb. 2. Ch. 3.}
\]
or \[
x = 1 \quad s - \frac{a}{x^n} = s \times - \frac{a}{x} \text{ Whence follows,}
\]

Probl.
<table>
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<th>Given</th>
<th>Reqd.</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$a, x, n.$</td>
<td>$s.$</td>
<td>$\frac{a}{x^a - x - 1}$</td>
</tr>
<tr>
<td>2.</td>
<td>$s, x, n.$</td>
<td>$a.$</td>
<td>$\frac{x^n \times x - 1 \times s}{x^n - 1}$</td>
</tr>
<tr>
<td>3.</td>
<td>$s, x, a.$</td>
<td>$n.$</td>
<td>$\frac{L, a - L, a + s - s \times s}{L, x}$</td>
</tr>
<tr>
<td>4.</td>
<td>$s, a, n.$</td>
<td>$x.$</td>
<td>$\frac{a + s}{s} \times s = \frac{a}{s}$</td>
</tr>
</tbody>
</table>

**Quest. 1.** The Annuity $a$ is to be sold for $n$ Years, allowing the Purchaser the Rate $r$; what is the Present worth $s$ of the Annuity?

**Quest. 2.** Having the Sum $s$ ready to be laid out, at the Rate $r$, to buy an Annuity for $n$ Years; what Annuity will it Purchase?

**Quest. 3.** The Annuity $a$ is to be sold for $n$ Years, for the ready Sum $s$; what rate $r$ has the Purchaser for his Money?

**Quest. 4.** The Annuity $a$ is made over for Payment of a Debt $s$, allowing the Creditor the Rate of Interest $r$; in what Time $n$ will the Debt be paid?

Several other more Practical Rules for solving the last Problem in this, and the former Case, are easily found; we shall shew hereafter.

**Corollary 1.**

By Supposing $n$, in the last Theorems, to be Infinite, and $a$ the Annual Rent; it follows that $s = s \times - a$.

Whence, Rules are drawn for Buying and Selling of Estates in Fee-Simple, allowing Comm. Interest.
QUEST. 1. There is a Fee-Simple to be sold, of a l. per Annum, allowing the Purchaser the Rate r, Comp. Interest; what Sum s will Purchase this Estate?

QUEST. 2. There is a Sum s, ready to be laid out, at the Rate r, Comp. Interest, for Buying a Fee-Simple; what Yearly Rent can such an Estate be of?

QUEST. 3. Having with the Sum s bought a Fee-Simple, whose Yearly Value is a l. what Rate r of Comp. Interest was allow'd for the Money?

COROLLARY 2.

Whence also, If it be required, how many Years Purchase, any Annuity is worth, allowing Comp. Int. Suppose N the Number of Years sought;

Now, that \( N = \frac{s}{x - s} \) is evident; and \( a = s \times - s \) by Rule.

Theref. \( N = \frac{s}{s \times - s} = \frac{1}{x - 1} \) by Division; hence,

<table>
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<th>Req'd.</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( x )</td>
<td>( N )</td>
<td>( \frac{1}{x - 1} )</td>
</tr>
<tr>
<td>2.</td>
<td>( N )</td>
<td>( x )</td>
<td>( N + 1 : N )</td>
</tr>
</tbody>
</table>
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Quest. 1. There is a Fee-Simple to be sold; how many Years Purchase N is it worth, allowing the Purchaser the Rate of Interest?

Quest. 2. There is a Fee-Simple bought, and N Years Purchase given for it; what Rate of Com. Interest is the Money valued at?

CHAP. XI.

I. Of Combinations of Quantities:

Definition.

The several Ways or Different Cases, of Taking or Leaving any Number of Quantities, out of any Number of Things exposed, without regarding their Order or Places, are called the Combinations of Quantities.

Let N be the Number of Things exposed.

n the Number of Quantities to be taken or left.

1. The Combinations of 2 Quantities, in these 6, viz. a, b, c, d, e, f, are 15.

\[
\begin{align*}
\{ a & \text{ --- } ab, \ a c, \ a d, \ a e, \ a f; \ 5. \\
b & \text{ --- } bc, \ bd, \ be, \ bf; \ 4. \\
c & \text{ --- } cd, \ ce, \ cf; \ 3. \\
d & \text{ --- } de, \ df; \ 2. \\
e & \text{ --- } ef; \ 1.
\end{align*}
\]

Therefore the Combinations of 2 in 6 is 15;

which is a Figurate Number of the 3d. Order, whose Side is
is \( N - 1 \), or \( N - n - 1 \); Because ’tis the Aggregate of a Series of the 2d. Order.

2. The Combinations of 3 Quantities in these 6, are 20, \( \text{viz.} \)

\[
\begin{align*}
& abc, a bd, a be, a bf; 4. b cd, b ce, b cf; 3. c de, c df; 2. d ef; 1. \\
& ace, af e; 3. b de, b df; 2. c ef; 1. \\
& ade, a df; 2. bef; 1. \\
& aef; 1 \\
& 10
\end{align*}
\]

And \( 10 + 6 + 3 + 1 = 20 \); which is a Figurate Number of the 4th Order, whose side is \( N - 2 \), or \( N - n - 1 \);

Because ’tis the Aggregate of a Series of the 3d Order; The same in any other.

Hence, \( n + 1 \) expresses what Order of Figurate Numbers to take.

If \( n = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \\ \text{&c.} \end{cases} \) then \( \begin{cases} 1 = s, \text{or Side of the Fig. Number.} \\ N - 0 \\ N - 1 \\ N - 2 \\ \text{&c.} \end{cases} = s = N - n - 1. \)

Consequently, \( \frac{s + 0}{1} \times \frac{s + 1}{2} \times \frac{s + 2}{3} \text{&c. to } \frac{s + n - 1}{n} \) shall be the Figurate Number required, or the Number of Combinations of \( n \) Quantities in \( N \). by Prob. 4. Cb. 8.

\( S C H O L I V U M. \)

Whence, the usual Examples, given by Writers on this Subject, are easily Solv’d; and the Reason of those Solutions made evident.
II. Of Elections of Quantities.

The Sum of all the Combinations found by taking 0, 1, 2, 3, &c. Quantities out of any Number of Things exposed, is called the Election of Quantities.

Thus in 6 Quantities, the Elections are 64.

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
\end{array}
\]

The Combinations of 3 in 6, is

\[
\begin{array}{cccc}
1 & 6 & 6 & 1 \\
6 & 20 & 15 & 6 \\
20 & 15 & 6 & 1 \\
\end{array}
\]

Therefore the Election of 6 Quantities is 64: And so in any other Number.

Hence 'tis evident, that the several Combinations in any Number, are only the Uncia of the several Terms of a Binomial Root, rais'd to the Power N; or the Members of the respective Powers of \(1 \div 1\) taken as a Binomial:

Therefore \(\frac{1}{1+i}^N\) or \(2^N\) is the Election of N Quantities, or the Number of Varieties in taking or leaving, 0, 1, 2, 3, &c. of any Number of Things expos'd, without regarding their Order.

Note, If the taking of 0 be excluded; Then \(2^N - 1\) will be Election of N Quantities.

But if the taking of 0 and 1 be excluded; Then \(2^N - N+1\) will be the Elections of N Quantities.
III. Of Permutations of Quantities.

**Definition.**

The several ways, that the Order of a certain Number of Different Quantities, may be varied or differently Placed, are called Alterations, Variations, or Permutations of Quantities.

As suppose the Quantities were a, b, c, d, &c. 'Tis plain, In 1 Quant. a. In 2 Quant. a, b. In 3 Quant. a, b, c, &c.

\[
\begin{align*}
\{a, b, c\} & \quad \{a, b, c\} \\
1 &= 1 \text{ var.} & 1 \times 2 &= 2 \text{ variat.} \\
\{a, b, c\} & \quad \{a, b, c\} \\
1 \times 2 \times 3 &= 6 \text{ Var.}
\end{align*}
\]

Thereof. N \times N \times N \times \ldots \times N = 1 \times N \times 2 \times N \times \ldots = N^N \text{, &c. to } N - N

is the Number of Alterations, or Changes of Order in N Quantities.

1. Hence the Changes that any Number of Bells admit of, are readily found.

2. As also, how many ways, the Letters of a Name, or Word may be differently disposed of, by way of Anagram; As the word Roma admits of \(4 \times 3 \times 2 \times 1 = 24\) Changes.

But if one, or more of the Letters, do occur more than once; Then the whole Number of Changes, must be divided by the Product of the Changes of the Repetition of those Letters; Thus,

\[
\text{a a a b c: admits of } \frac{5 \times 4 \times 3 \times 2 \times 1}{6 \times 2} = \frac{120}{12} = 10 \text{ Changes.}
\]

IV. Com-
IV. Composition of Quantities.

Definition.

The several Dispositions of any Number of Quantities placed according to all the different ways, they are capable of, by taking them either singly, or connecting the given Rank to each Term therein, as often as required, are call'd Composition of Quantities: Thus,

The Composition of 2 Quantities in these $3^0 a, b, c$, is 9.

For with $a$, $b$, $c$.

$\begin{array}{ccc}
  aa & ab & ac \\
  ba & bb & bc \\
  ca & cb & cc \\
\end{array}$

$3 + 3 + 3 = 9$

And the Composition of 3 Quantities in Three is 27, or Generally $N^n$ is the Composition of $n$ Quantities in $N$.

Corollary.

Whence because $N^1$, $N^2$, $N^3$, $N^4$, &c. to $N^n$ or $NN$ is a Geometric Progression, whose First Term = $N$ = Ratio, and Number of Terms = $n$. Therefore

its Sum $(N \times \frac{N^n - 1}{N - 1} = \frac{NN^n - N}{N - 1} = )N^n - 1 + \frac{N^n - 1}{N - 1}$

(by Prop. 1, Ch. 9.) will give the several Dispositions there may be of $N$ Quantities, taking them one and one, two and two, &c. to $N$ Quantities.
And since \( 24^{24} \) \((N^N) = 133373577685028412444908472843776\) is the Number of the several Compositions that 24 Letters make, by taking each time 24; therefore,

\[
\frac{N^N - 1 + N^{N-1}}{N - 1} = 13917242888897392999425128493402200
\]

is the Number of all the words significant and insignificant, that can be made of 24 Letters.

Variety of Instances might have been given on the several Heads of this First Part, but because the Application is endless, the Learner (in most Places) is left to his Liberty of taking what Examples he pleases: For we would use as much brevity as possible, in Order that other useful things design'd for this Treatise, may come in: And 'tis hoped what has already been laid down, if well regarded, is sufficient without any farther Illustration, to render the whole Art of Numbering, whether Theoretical or Practical, intelligible to any reasonable Capacity.

The Symbols used in the following Part, are only the Common ones, as \( L \), for Angle; \( \perp \), for Right Angle; \( L \) for Perpendicular; \( \parallel \), for Parallel; \( \infty \), for Infinite: with other Abbreviations of words, which the Reader will easily understand.
SYNOPSIS
Palmariorum Matheos.
PART II.
Containing the
PRINCIPLES OF
GEOMETRY.

1. That science by which we learn how to compare things extended the one with the other, or determine how the one is more less, or as much extended as another, is called GEOMETRY.

2. In things extended we distinguish three dimensions, viz. Length, Breadth, and Thickness.
The Quantity wherein we consider only Length, and Breadth, we consider nothing but Length and Surface.

And that wherein we consider neither Length, Breadth, or Thickness, and therefore is conceiv'd to have no Parts, or whose Divisibility is not consider'd, is called a Point.

3. Lines, Surfaces, and Solids may be consider'd as generated by a continual Motion of Points, Lines, and Surfaces: And as these Motions may be different, so may there be different Sorts of Lines, Surfaces, and Solids described by them.

4. If a Point move towards another the nearest way, it will describe a Right Line, otherwise a Curved one: So a Right Line moved on a Right, or Curved Line, will produce a Plane, or Curved Surface: The same may be conceived of Solids.

5. The Curve-line whose parts are equally distant from a Point in the same Plane, is called a Circumference; that Point the Centre; Right Lines drawn from the Centre to the Circumference are called Radii; Those drawn from one Point of the Circumference to the other are called Chords; and that Chord which passes thro' the Centre is called a Diameter and divides the Circumference equally in two. Also any Circumference may be conceived to be divided into 360 equal parts called Degrees, and each Degree into 60 equal parts, called Minutes, and each of those again subdivided into 60 Seconds, &c. Any part of the Circumference is called an Arc.

6. The Inclination of two Right Lines meeting in a Point, to as not to make one Right Line, is called a Plane Angle; which therefore may be conceived to be generated by the Rotation of Sides: And its Measure is an Arc described from the Angular Point as a Centre, and intercepted between the Lines which form it. An Angle is said to be Equal to, Greater, or Less than another, according as the Arc which measures it contains, as Many, More, or Fewer of the equal parts into which that Circumference is supposed to be divided.

7. When
7. When a Right Line stands upon another, so as to make equal Inclinations or Angles on each Side thereof, that Line is called a Perpendicular; and is the nearest Distance between a given Point and a given Line; those equal Angles are called Right Angles; which therefore have each for their Measure of the Circumference or 90 Deg. Conseq. the Circums. = two Right Angles; and the whole Circums. = 4 Ls. An Angle greater, or less than a Right Angle is called Obuse, or Acute.

8. Half the Chord of twice any Arc is called the Right Sine of that Arc: That part intercepted between the Arc and its Right Sine, is called the Versed Sine thereof. And the Sine of an Angle is the Sine of the Arc which measures that Angle; and is a Perpendicular from one end of the Arc, on a Right Line drawn from the Angular Point to the other.

9. A Right Line falling upon another makes two Angles equal to two Right Angles. For they are measured by of the Circumference, which measures 2 Ls (by 7.) Consequently all the Ls made at the same Points, on the same side of a Right Line, are = 2 Ls. Therefore, that is a Right Line on which another falling makes two Ls = to 2 Ls. And all the Ls that can be made about any Points are = 4 Ls.

10. Therefore the op. Ls (a, c) of crossing Lines are =; For a - b = 2 Ls = b - c (by 9) therefore a = c (by Subduction.)

11. A Tangent to any Point of the Circumference is Perpendicular to the Radius drawn to that Point. For the Radius is the nearest Distance.

12. A Right Line drawn Perpendicular to the Radius, passing thro’ one end of an Arc, and limited by a Line called Secant, drawn from the Centre thro’ the other end, is the Tangent of that Arc.

13. The Radius being supposed divided into any Number of parts; the Quantity of the Sines, Tangents, &c. of all Angles are estimated according as they contain more or fewer of such Parts. Therefore,
In the same or equal Circumferences, the Sines, Tangents, &c. of equal Arcs or Angles are respectively equal, because they consist of an equal Number of Parts of the same or equal Radii.

In unequal Circumferences, the Sines, Tangents, &c. of equal arcs, or Similar Arcs are similar also, because they contain the like Number of Parts, of their respective Radii.

14. The Difference of an Arc from \( \frac{1}{2} \), or \( \frac{1}{3} \) the Circumference is called the Supplement, or Complement of that Arc. An Arc and its Supplement, have the Same Sine, Tangent, and Secant. And the Sine, Tangent, or Secant of the Complement, is called the Co-Sine, Co-Tangent, or Co-Secant of the Arc, whole Complement that is.

15. Lines which no where incline one to the other are called Parallel Lines.

16. And two Parallel, right Lines (p, p) have the same Inclination to a third Line (L). For Lines parallel may be taken as one broad Line.

17. If Parallel Right Lines be cut by a Right Line;

1. All the op. l's are equal; And the contrary. For \( a = c, e = g \) (by 16.), but \( a = b, c = d, e = f, g = h \) (by 10.). Therefore \( a = b = e = d \); and \( e = f = g = h \) by Substitution.

2. The two op. l's, whether internal or external are \( 2 \) \( \perp \) l's; and the contrary. For \( f + a = b + e = c + b = d + g = 2 \) \( \perp \) l's (by p); but \( a = c, e = g \) (by 16.). Therefore \( f + c = b + g = a + b = e + d = 2 \) \( \perp \) l's by Substitution.

18. And, all Chords drawn parallel to the Tangents, are bisected by the Diameter passing thro' the Point of Contact. Because they are perpendicular to it (by 12.) and their Extremities are equally distant from the Centre, and also from the Point of Contact; therefore the Parts of the Chords will be Sines of equal Arcs, and conseq. equal (by 13.).

Therefore if a Chord be bisected by another at Right Angles, the bisecting one is a Diameter. And the contrary.

19. Whence
19. *When a Circumference may be drawn thro' any three Points, not in a Right Line*; and the Intersection of two Right Lines bisecting the Distance between those Points at L's is the Centre; *Since Right Lines bisecting Chords at L's are Diameters*; (by r8.)

20. *A bounded Space is called a Figure*; if bounded by Right, or Curv'd Lines; 'tis called a Rectilinear, or Curvilinear Plane Figure.

21. *Those Rectilinear Plane Figures, whose L's are respectively equal, and the Sides about those L's directly proportional, are called Similar Rectilinear Figures.*

22. *Those Rectilinear Plane Figures that are bounded by 3, or 4 Right Lines, are called Triangular, Quadrilateral or Quadrangular Plane Figures.*

23. *Triangular Figures are considered with respect to their Sides or Angles.*

24. *Those that have 3 equal, 2 equal, or 3, unequal Sides are called Equilateral, Isosceles, or Scalene Triangles.*

25. *Those that have 1 Right L, 1 Oblique, or 3 Acute L's are called Right, Obtuse, or Acute L'd Δ's.*

26. *Quadrilateral Figures, whose opposite Sides are parallel, are called Parallelograms. And,*

Right L'd Δ's all 2 Sides equal 2 Squares.
Pгр. having 2 only op. 5 are call'd 2 Rectangles.

AmOblique L'd Δ's all 2 Sides equal 2 Rhombi.
Pгр. having 2 only op. 5 is call'd a 2 Rhomboids.

27. *And a Parallelogram may be conceived to be generated by a Right moveable Line drawn uniformly into the Length of a Right Immoveable one: Therefore the Surface generated by those Lines shall contain so many little Planes equiangular with the whole, as there are Unis in the Product of their Parts.*

28. *Therefore Pгрs. and consequently Δ's having one L equal, are as the Product of the Sides about that L. And if those Sides be reciprocally :: 1, the Parallelograms, or Δ's shall be equal, and the contrary.*

29. *Also*
27. Also, if \( h \) and \( b \) be the Height and Breadth of a Rectangle; its Area will be \( h \cdot b \), from its Genesis, i.e., will contain so many square Areas as there are Units in the Product of its Height and Breadth.

And since the Contents of all Surfaces are estimated by the Square of some known Length; therefore such Figures must be reduced to Rectangular ones, before they can be Measured.

28. A Quadrilateral Figure of unequal Sides is called a Trapezium. And all other Rectilineal Figures, are in general, called Polygons, and have particular Names from the Number of their Sides or Angles, as those of 5 Sides Pentagons, of 6, Hexagons, of 7, Heptagons, &c.

All Plane Figures having equal, or unequal Sides or Angles, are called Regular, or Irregular Figures.

29. A Right Line drawn from \( L \) to \( L \) in any Figure, is called a Diagonal, or by some a Diameter.

The Height of any Figure is the nearest Distance between its Top and Base: Therefore all Figures between the same Parallels have the same Height.

30. A Plane Figure whose Extremities are equally distant from a Point therein, is called a Circle: And may be conceived to the generated by the Revolution of a right Line, one end being fix’d as the Centre, the other describing the Circumference of the Circle. And the Space, less than a Semi-Circle, described by a partial Converion is called the Sector of a Circle. A Part of a Circle cut off by a Right Line is called the Segment of a Circle.

31. Hence, all Circumferences, as also like Arcs, their Sines, Tangents, &c. are as their Radii.

32. The Genesis of Solids may be exhibited various ways; some respecting the Dimension both of their Soli- dity and Surface; others, that of the Surface only. As,

1. A Parallelogram being conceived to move uniformly the Length of an inmoveable Right Line, shall generate a Solid called a Parallelepiped: Or if the Describing Surface be a Square moving the Length of, and perpendicular to, its Side, the Solid is called a Cube: Therefore a Rectangular Parallelepiped contains so many equal Cubes, as there are Units in the Product of the Square Areas in
in the describing plane, by the parts, of like measure for Length, in the Length moved: so that if $b$ be its Height, and $b$ the Area of its Base, then $bb = \text{Solid Content}$; or if $l,b$ be the Length, and Breadth of the Base, then $bbl = \text{Solidity.}$

If the describing Plane be a Polygon, or Circle, the generated Solid is called a Prism, or Cylinder.

But if the describing Polygonal, or Circular Plane, in moving be supposed to decrease uniformly, till it comes to a Point, the Solid generated is called a Pyramid, or Cone.

2. A Right Line moving uniformly, so that its ends may describe the Periphery of two parallel, similar, and equal Polygons, will generate the Surface of a Prism, whose Bases are those Polygons.

If the Bases are Prs. $\Delta s$, or Circles, the Surface generated will be that of a Parallelepiped, Triangular Prism, or Cylinder.

The right Line connecting the Centres of the Polygonal Bases is called the Axis.

And a right Line fix'd by one end to a Point, and with the other describing a Polygon, or Circle, will generate the Surface of a Pyramid, or Cone, whose Base is that Polygon, or Circle.

The right Line connecting the fix'd Point, and the Centre of the Base is also called the Axis.

And that Solid whose Axe is perpendicular, or oblique to its Base is said to be Right, or Scalenous.

The Conversion of a Semicircle round the Diameter will generate a Solid called a Sphere.

Solids contain'd under an equal Number of like Surfaces, are said to be Similar.

33. And Quantities, as also their Ratio's, that continually tend to an Equality, and therefore that approach nearer the one to the other, than any Difference that can possibly be assign'd, do at last become equal. For, either they'll be at last equal, or there will be some Difference ($d$) nearer than which they cannot approach; But they do continu-
ally approach by Supposition; therefore there can at last be no difference, therefore they must be equal.

Hence, all Curved Lines may be considered as composed of an Infinite Number of Infinitely little right Lines: And any one of them Produced, only Touches the Curve, therefore is called a Tangent to that Point of the Curve.

And as all Surfaces may be consider'd as composed of an Infinite Number of Parallel, Right, or Curved Lines, or Surfaces of an infinitely small Breadth. So also Solids, of an Infinite Number of Parallel, Plane, or Curved Surfaces, or Solids of an infinitely small Thickness, called the Elements of Figures.

34. And by considering Quantities as generated by continual Motion, 'tis apparent, that in equal Spaces of Time, they will become greater, or less proportionally as the Celerity of the Motion by which they are so generated is greater or less: Hence the Celerity of the Motion is very properly called Fluxion, and the Quantity generated Fluent.

Now these Fluxions of Quantities are in the First Ratio of their Nascent Augments; and may be express'd by Finite Quantities proportional to them.

And as \( x \) (by moving uniformly) becomes \( x + \frac{dx}{n} \),
\[ x^n \text{ becomes } x^n + \frac{n x^{n-1}}{n} \times 0 + \frac{1}{n} n x^n - n \]
\[ x^{n-1} \times 0 + \frac{1}{n} n x^{n-2} \]
But the Augments \( 0, \frac{1}{n} n x^{n-2} \) &c.

are as \( 1 \) and \( n x^{n-1} \) as \( 1 \) & \( n x^{n-1} \) &c.

Th. Fl. \( x \cdot F, x^n : : 1 ; n x^n - 1 \)

The Flux. of the Fluent \( x, x, x, \&c. \), are denoted by \( x, x, x, \&c. \), which are the \( 1st, 2d, 3d, 4th, \&c. \) Flux. of \( x \).

Also, as \( x, \frac{dx}{F}, \&c. \), and \( \frac{dx}{x} = F, x \); So we may consider
sider \( x = F, x, \) and \( x = F, x', \) &c. As also denote the Fluxion of

\[
\frac{a}{a-x-x^2} \frac{1}{x+x}, \frac{a}{a-x-x^2} \frac{1}{x+x}, \frac{a}{a-x-x^2} \frac{1}{x+x}
\]

by \( a \frac{x}{x-x^2} \frac{1}{x+x}, a \frac{x}{x-x^2} \frac{1}{x+x}, a \frac{x}{x-x^2} \frac{1}{x+x}, \frac{a}{a-x-x^2} \frac{1}{x+x} \)

Hence \( n x^{n-1} : x : n x^{n-1} = F, x^n, \) whether \( n \) be Affirmative or Negative, Integer or Fraction, as for Instance.

\[
F, x^{-n} \left( \frac{1}{x^n} \right) = \left( -n x x^{-n} = -n x^2 x^{-n} = \right) \frac{n}{n x^{n+1}}
\]

Th. \( F, \left( \frac{x^{-1}}{x^n} \right) = \left( \frac{x}{x^n} = \right) \frac{x}{x^n} \)

\[
F, \frac{1}{x} \left( \frac{x^{-1}}{x^n} \right) = \frac{m}{m} \frac{x}{x} - \frac{m}{m} \frac{x}{x}
\]

\[
F, x^{m} = \left( \frac{1}{2} x^{m-1} = \frac{1}{2} x^{m-1} = \frac{1}{2} x^{m-1} = \right) \frac{m}{2 x^m}
\]

35. In any given Equation, involving Fluents, to find the Fluxion.

Rule. Multiply every Term of the Equation separately by the several Indices of the Powers of the Fluents therein; and in every such Product, change one of the Roots of the Powers into its Fluxion, the Aggregate of all the Products, connected under their proper Signs, will be the Fluxion of the Equation sought.

This Rule is usually demonstrated after this manner;

If \( x^3 - x y^2 = a^2 \) \( = b^2 = o; \) Then its Fluxion is

\[
3 x^{\frac{3}{2}} x^{\frac{1}{2}} - 2 x y \frac{1}{y^2} = 2 x y = o; \quad \text{For supp. o an}
\]

G g 2

Inf-
Infinitely small Quantity; let \( o \dot{x} \), \( o \dot{y} \), \( o \dot{z} \), represent the Instantaneous Increments of the Fluents \( x, y, z \); These after a Momentary Increment of Time, will become

\[
x + o \dot{x}, y + o \dot{y}, z + o \dot{z},
\]

which being substituted instead of \( x, y, z \), in the given Equation, it will be \( x^3 + 3x^2o \dot{x} + 3x o^2 \dot{x}^2 + o^3 \dot{x}^3 - x y^2 - o \dot{y} x^2 - 2x o y \dot{y} - 2x o^2 \dot{y}^2 - x o^3 \dot{y}^3 + a^2 \dot{z} = b^3 = 0 
\)

From this Subduct the given Equation; Then divide the Remainder by \( o \); and reject all Terms multiplied by it, (as being infinitely little;) There results

\[
3x^2 \dot{x} - x y^2 - 2x y \dot{y} + a^2 \dot{z} = 0
\]

Also, if \( x^3 - x y^2 - a^2 \dot{x} x - x y^2 \frac{1}{2} - b^3 = 0 \), then

its Fluxion is \( 3x^2 \dot{x} - x y^2 - 2x y \dot{y} + a^2 \dot{x} x - x y \frac{1}{2} = 0 \).

Or let \( z = \frac{a x - y y}{x} \), th. \( z^2 = a x - y^2 \); and (by this) \( 2z \dot{z} = a \dot{x} - 2y \dot{y} \), th. \( \dot{z} = \frac{a \dot{x} - 2y \dot{y}}{2z} \), i.e.

\[
\frac{2x - y^3}{2y a x - y y} = a \dot{x} - 2y \dot{y} \times \frac{1}{2y a x - y y}
\]

Th. \( 3x^2 \dot{x} - x y^2 - 2x y \dot{y} + a^2 \dot{z} - 2a^2 y \dot{y} = 0 \)

And by repeating the Operation, the Second, Third, &c. Fluxions of Equations are found.
If \( z^3 - 3z^2 + 3z - 3 = 0 \), then by operation

1. \( z^3 + 3z^2y - 4z = 0 \)

2. \( z^3 + 6z^2y^2 + 3z^2y^3 + 3z^2y^3 + y^4 - 4z^2z^2 = 0 \)

Hence, the Fluxion of Quantities multiplied, is the Sum of the Products of the Fluxion of each Factor, by the Product of the other Factors. Thus,

\[ F, \quad x \cdot y \text{ is } \frac{d}{dx}(xy) = x \frac{dy}{dx} + y \frac{dx}{dx} \]

\[ F, \quad 2x - x^2 \text{ is } \frac{d}{dx}(2x - x^2) = 2 \frac{d}{dx}x - x \frac{d}{dx}x \]

\[ F, \quad 3x + 2x^2 \text{ is } \frac{d}{dx}(3x + 2x^2) = 3 \frac{d}{dx}x + 2 \frac{d}{dx}x \]

\[ F, \quad 3ax - 2z^2 \text{ is } \frac{d}{dx}(3ax - 2z^2) = 3a \frac{d}{dx}x - 2 \frac{d}{dx}z^2 \]

and so in others. Also in Fractions

\[ F, \quad \frac{x}{y} \left( x + \frac{1}{y} \right) \text{ is } \frac{d}{dx} \left( \frac{x}{y} + \frac{1}{y} \right) = \frac{y \frac{d}{dx}x - x \frac{d}{dx}y}{y^2} \]

\[ F, \quad \frac{a}{x} \left( a \cdot \frac{1}{x} \right) \text{ is } \frac{d}{dx} \left( \frac{a}{x} \cdot \frac{1}{x} \right) = - \frac{a \frac{d}{dx}x}{x^2} \]

Tb. the Flux. of any Fraction, (N being Numerat. and D Denominat.) is \( \frac{ND - ND}{DD} \).

If the Indices are Flowing Quantities; as for instance,

\[ F, \quad y \text{ is } \frac{d}{dx}y = x^{x+1} + x \cdot x^{x+1} - x \]

\[ y + F, y \left( = y + \frac{d}{dx}y \right) = x^{x+1} + x^{x+1} - 1 \]

Therefore if the Index be the Sum, Product, or Power, of Fluent; 'tis but substituting it, and its Flux, instead of \( x \) and \( x \) in this.
36. And the Fluent \((\varphi)\) of \(x^{\frac{n}{r}}\) is \(\frac{x}{n+1}\).

The Fluent of \(ax^r + b + c\) is easily found to be

\[
\frac{1}{mn + r + 1} \times \frac{a}{c} (A) \times x^{r-n+1}
\]

\[
\frac{1}{mn + r - n + 1} \times \frac{ab}{c} (B) \times x^{r-2n+1}
\]

\[
\frac{2}{mn + r - 2n + 1} \times \frac{ab}{c} (C) \times x^{r-3n+1}
\]

\[
\text{Hence } \varphi, \text{ of } ay + ax - y^2 = 0 \text{ is } \frac{x}{2} \left( ay - y^2 \right)^{\frac{1}{2}}\]

And the like of others.

37. Where the Extreme Value, whether Greatest or Least, of any Quantity is required; since it is an Invariable by Sup. and its flux = 0; Therefore to determine it to an Extremum, put the Equation into Fluxions, let the Fluxion of that Quantity = 0; then will all the Terms wherein it is found vanish, and the Extremum be determined by the remaining ones.

38. The \(L (n,N)\) made by the Tangent (PG, PT) and Chord (PB) has for its measure an Arc = half that subsended by the Chord.

Draw a Diameter \(L PB\), then is the Arc \(a\), and Chord PB biseected (by 18), draw the Rad. CP, as also \(CD \parallel PB\).

Then \(L n + L z = L = L r + L x\) (by Confr.) But \(z = x\) (by 17) then \(n = r\) by Eq. Subduæ. And \(\frac{1}{2} \) a Measures the \(L r\) (by 6) therefore also the \(L n\). Also \(n + N\) are measured by \(\frac{1}{2} a + \frac{1}{2} A\) (by 9) th. N is measured by \(\frac{1}{2} A\).

39. An \(L (c)\) in the Circumference made by two Chords, is measured by \(\frac{1}{2}\) its subsending Arc \(A\).
Chap. 11. Palmariorum Matheseos.

For $\frac{1}{2} N + A + M$ measures $n + c + m$
(by 9) But $\frac{1}{2} N + M$ measures $n, m$ (by 38)
Therefore $\frac{1}{2} A$ measures $c$.

40. All $L$s in the Circumference, subtended by the same Arc or Chord, are equal. For each is measured by $\frac{1}{2}$ the subtending Arc.

41. An $L$ (C) at the Centre is double that (c) at the Circumference standing on the same Arc (A.) For C is measured by the whole (by 6), and $c$ by $\frac{1}{2}$ the Arc A (by 39).

42. An $L$ in a Segment $>\frac{1}{2}, <$ Semicircle is Acute, Right, Obtuse: For $'tis$ subtended by an Arc $<, =$, $>$. Semicircumference, whose $\frac{1}{2}$ (the Measure of the $L$) by 39 is $<, =$, $> 90^\circ$, and therefore is Acute, Right, Obtuse (by 7).

43. Whence, To draw a Tangent (PT) to any points (P) of the Circumference of a Circle: Let PB = Rad. (PC), draw CB out, let BT = BC, draw TP the Tangent sought.

For since PB = BC = BT (by Constr.) an Arc passing thro' C, P, T, (by 19) will be a $\frac{1}{2}$ Circumference, th. $LP = L$, (by 42) th.
PT is a Tangent to the point P (by 12).

Hence the Practical Methods of Erecting and Letting fall Perpendiculars are evident.

44. Opposite $L$s ($n + N$) in the Circumference, standing on the same Chord are $= 2 L$s. For each is measured by $\frac{1}{2}$ its subtending Arc, therefore both by $\frac{1}{2}$ the Circumference, which measures $2 L$s (by 7).

45. Hence, The op. $L$s of a Quadrilateral Fig. iner. in a Circle are $= 2 L$s. And that Quadrilat. Fig. the Sum of whose op. $L$s is $= 2 L$s, may be inscrib'd in a Circle.

46. And the Sides thereof be produc'd, the external $L$ (a) is $= to$ the internal op. one (N.) For $a (+ n = 2 L$ (by 9) = N $(+ n$ by this.)

47. The $L$ $= x, z$ made by a Tangent and Chord, is $= to an L$ $(u, o)$ in the op. Segment.

For $x, z = \frac{1}{2} a, \frac{1}{2} A$ (by 38) = $u, o$
(by 39)

48. Hence
48. Hence, To cut off a Segment from a given Circle capable of containing a given L (2).

Draw the Tangent $ab$ (by 43), make $x = L \gamma$, then $a = x$ (by 47) = $\gamma$, therefore $tae$ is the Segment reqd.

49. Also, having the Chord $ab$ of a Segment capable of containing a given L ($x$): To find the Point ($p$) thro' which the Arc shall pass.

Draw $ae$, and at any point ($e$) make $Le = Lx$, draw $bp \parallel de$, then $q (== e) = x$, therefore the Arc shall pass thro' $p$. In the same manner the other points are found.

50. In a Circle, equal Chords are equally distant from the Centre; and the Contrary. For those = Chords are = double Sines, therefore their Co-sines, which is their distance from the Centre, must be =.

51. In a Triangle, the $\frac{1}{2}$ of each Side is the Sine of its op. L. For (being inscr. in a Circle by 19) the Sides are Chords of Arcs, which measure $\frac{1}{2}$ their op. Ls; And $\frac{1}{2}$ those Chords are the Sines of those Arcs (by 8) i.e. $\frac{1}{2}$ the Sides are the Sines of their op. Ls.

52. The 3 Ls of every Plane $\triangle$, is $= 2 \perp s$: For (being inscr. in a Circle by 19) their Measure is $\frac{1}{2}$ the Circumference, which measure $2 \perp s$ (by 7).

53. Therefore in a Plane $\triangle$, if the Side be produced, the extern. L ($d$) will be = to the two intern. and op. ones ($a + b$.) For $a + b (\perp e = 2 \perp s$ by 52) = $d (\perp c$ by 9).

54. Hence the measure of an Angle, as $c$ in Figure

1. Neither at Centre or Circumf. made by two Crossing Chords; Or.

2. In the Circumf. made by the Chord and Secant; is ($= y \perp x$ by 53) = $\frac{1}{2}$ ($A + \frac{1}{2}a$ (by 39)).

3. With-
3. Without the Circumference, made by two Secants; or
4. Made by two Tangents; is \( \frac{x - y}{2a} \) (by 53) = \( \frac{1}{2} A - \frac{1}{2} a \) (by 39)

55. The mixt \( L \) RPd made by the Circumference, and Rad. is greater than any Reillinear Acute \( L \) (RPy or z.) For drawing RN Perp. PY, then \( LN \) (= \( \perp \)) > \( LP \); therefore RP > RN, and the point N is within the Circle Conseq. \( \perp z < L \) RPd.

Whence, the \( L \) made by the Circumference and Tangent is less than any Acute \( L \).

Therefore no Right Line can be drawn between the Tangent and Circumference, so an infinite Number of Curv'd ones may.

56. A Curvilinear \( L \) (CPC.) made by two Intersecting Circumferences (C, c) is \( = \) to a Reillinear \( L \) (RPPr) made by the Radii (RP, rP) drawn to the Intersect. point (P.) For, drawing the Tangents PT, Pr, then \( L \)PC \( = \) LPT, and rPT. (\( + TP = \perp \)) = RPr (\( + rPT \)).

57. In a Triangle \( \perp \), \( \perp \), \( \perp \) Angles are subtended by \( \perp \), \( \perp \), \( \perp \) Sides; And the contrary. For \( \perp \), \( \perp \), \( \perp \) Sides are Chords of Arcs which measure \( \perp \), \( \perp \), \( \perp \) Angles.

1. Therefore Equilateral \( \triangle \)s must be also Equiangular; And the contrary.

2. And those \( \triangle \)s are equal, if in both, either 3 Sides; or 2 Sides, with the included \( L \); or 2 Sides, with the \( L \) op. to the same Side; or 2 \( \triangle \), with the adjacent Side; or 2 \( \triangle \), with the Side subtending the same \( L \); be equal.

3. Therefore the \( L \)s as the Base of an Isosceles \( \triangle \) are \( = \); For the equal Sides are Chords of Arcs measuring \( = \) \( L \)s.

4. And that Line from the Vertex intersecting the Base of an Isosceles \( \triangle \), is Perp. to it.

58. The Intern. \( L \)s of any Polygon are \( = \) to twice as many \( L \)s as it has Sides, except 4. For (\( n \) being the No. of Sides or \( L \)s,) every Polygon may be divided into \( n - 2 \) \( \triangle \), (by Lines drawn from any \( L \) to all the reft, except the adjoining ones,) therefore will have \( 2n - 4 \) \( L \)s.
1. And (producing the Sides) all the extern \( L_s = 4 \times L_s \).
For each Intern with its Extern is \( = 2 \times L_s \), (by 9) therefore all the Intern and Extern \( L_s = 2n \times L_s \); but the Intern \( L_s = 2n - 4 \times L_s \) (by 58) the extern \( L_s = 4 \times L_s \).

2. Therefore the Intern \( L \) (A) of any Regular Polygon

\[ \frac{2n}{n} \times L_s = A \]

3. And the \( L \) (a) at the Centre is \( = 4 \times L_s \times \frac{2}{n} \) (\( = 2 \times L_s - A \))

59. Hence it follows, that there are only 3 Regular Surfaces that can fill a Space, viz. Triangles, Squares, and Hexagons.

Since all the \( L_s \) round a Point \( = 4 \times L_s \), and the \( L \) of

- Regular \( \Delta \) is \( \frac{1}{2} \times L_s \)
- Square is \( 1 \times L_s \)
- Hexagon is \( \frac{1}{3} \times L_s \)

Which multiply by \( 4 \) \( = 4 \times L_s \).

But in a Pentagon, \( 3L_s \) are less, \( 4L_s \) greater \( 4L_s \), and in Figures having more Sides, \( 4L_s \) greater \( 4L_s \), therefore, \&c.

60. Also, There are only 5 Regular Bodies. For a Solid \( L \) consists of less than \( 4 \times L_s \), and 3 Plane \( L_s \) are the fewest that can make it; But 6, 4, 3 \( L_s \) of \( \Delta \), \( \boxed{\square} \), Hexagons, make \( 4 \times L_s \), and \( 4L_s \) of Pentagons are greater \( 4L_s \), therefore only a \( \Delta \), \( \boxed{\square} \), and a Pentagon can form a Solid \( \boxed{5} \); so that there can be but 5 Regular Bodies, viz. the

<table>
<thead>
<tr>
<th>Tetrahedron</th>
<th>Octahedron</th>
<th>Icosahedron</th>
<th>Hexahedron</th>
<th>Dodecahedron</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4 \times L_s )</td>
<td>( 6 \times L_s )</td>
<td>( 12 \times L_s )</td>
<td>( 8 \times L_s )</td>
<td>( 20 \times L_s )</td>
</tr>
<tr>
<td>( 3 \times \Delta )</td>
<td>( 4 \times \Delta )</td>
<td>( 5 \times \Delta )</td>
<td>( 3 \times \boxed{\square} )</td>
<td>( 3 \times \boxed{\square} )</td>
</tr>
<tr>
<td>And has:</td>
<td>And has:</td>
<td>And has:</td>
<td>And has:</td>
<td>And has:</td>
</tr>
<tr>
<td>( 4 \times ) Plane sides</td>
<td>( 6 \times ) Plane sides</td>
<td>( 12 \times ) Plane sides</td>
<td>( 20 \times ) Plane sides</td>
<td>( 30 \times ) Plane sides</td>
</tr>
<tr>
<td>Line Edge</td>
<td>Line Edge</td>
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<tr>
<td>( 22 \times L_s )</td>
<td>( 12 \times L_s )</td>
<td>( 24 \times L_s )</td>
<td>( 24 \times L_s )</td>
<td>( 24 \times L_s )</td>
</tr>
</tbody>
</table>

61. The Intersection of 2 Planes is a Right Line: For the Right Line drawn on either Plane from any two Points of the Section will be the same. And

1. The Intersections of 2 Paral. Planes by a 3d are Paral.
2. A Perp. to crossing Lines, is Perp. to their Planes.

3. Right
3. Right Lines, or Planes, Perp. to the same Plane, or Right Lines are Parallel.
5. If a Right Line (L) be Perp. to any Plane (P) then all Planes passing thro' (L) are Perp. to P; And if 2 Planes be Perp. to P, their Intersection will also be Perp. to P.
6. In any Number of Planes passing thro' the Line L, Paral. to a certain Plane P, their Intersection with P are || to L, and to each other.
7. Two pair of meeting Lines respectively || in different Planes contain = L, and Paral. Planes.
8. In Equiangular Δs, the sides about = Ls are directly :: 1; And the conrr. For, being incir. in a Cir-, the sides will be Chords of Similar Arcs.
9. Therefore Equiangular Δs are Similar (by 21); Conseq. similar Δs have their sides about = Ls :: 1.
10. And therefore those Δs are also similar, which have an L =, and the sides about that L :: 1; Or which have 2 sides :: 1, an L op. to one of 'em =, and the other of the same kind.
11. Hence similar Δs have their Heights :: 1 to their Bases; For making a Side next the vertical L Radius, the Heights will be Sines of Similar Arcs, which are as the Radii (by 31) and those Radii as the Bases (by 52).
12. And a Right Line, or Plane cutting a Triangle, or Pyramid, either parallel or subcontrary to the Base, cuts off a Figure similar to the whole. For the L at the Top is common, the others are :: by Confr. therefore the Figures are similar.
13. Conseq. in a Δ, the parts of the sides cut by a Right Line drawn || to its Base are directly :: 1; And the conrr. For the Δ cut off being Sim- to the whole (by Confr.) the Sides about the common L at the Top will be :: 1 (by 62) therefore, Es.
14. Hence is drawn the Method of dividing a Right Line into a given :: 3, And of finding a 3d, or 4th :: 1 to 2, or 3 given right Lines; with Variety of Practices in Geometry.

H h 2 6. 18
Synopsis

Part 2.

6. In any $\triangle$, the Base is to its Parallel, as the Sides are to the Parals next the Vertex.

For $y : x :: n : m :: b : p$ (a) by 4. 63

$\frac{y}{a} : \frac{x}{b} : \frac{n}{m} : \frac{b}{p}$.  

7. In any Quadrilateral Fig. inscribed in a Circle, the product of the Diagonals is $= \text{the sum of the opposite sides.}$

Let $\angle DAE = \angle BAC$, $AC = y$, $BE = m$; then $\triangle$s $AED, ABC,$ and $ABE, ACD,$ are Similar.

Th. $\frac{b}{n} : \frac{m}{y} :: \frac{c}{d}$ by 63, th. $\frac{bc}{my} = \frac{ad}{y}$.

Th. $ad + bc = (my + ny) \frac{y}{x} m + n$.

8. All Figures of the same kind, standing upon the same, or equal Bases, and of $= \text{Altitudes are} =$. For the Elements of the one are respectively equal to those of the other, both in Number and Magnitude. Th. in Pgrs. Ppds. Prisms, and Cylinders, if $B$, $H$, be the Base, and $Height$, then $BH = \text{Consent}$. And in Triangles, the Area $= \frac{1}{2} BH$; bec., Triangle: Circumfer. Pgr. :: 1 : 2 (by this).

9. And a Rectilinear Figure of $n$ Sides may be reduced into $n - 2$ Triangles, $by n - 3$ Diagonals; and its Area is $= \text{the Area of all its Triangles.}$

10. Th. in a Circle whose Area call $A$, Circumf. $c$, Rad. $r$, since $\frac{1}{2} r^2 c (\frac{1}{2} rc \circ c) = A$, th. $\frac{1}{2} rc = A$, (by 36.) And $d^2 (4r) = \frac{1}{2} rc = 2r (d) = \frac{1}{2} c$. Also, if $c = \text{Arc}$, then the Sector $= \frac{1}{r} rc$.

11. All Homologous Figures (1) Of equal Bases are as their Heights, or of equal Heights, are as their Bases; (2) of different Bases and Heights, are as their Products, or those of the Sides into the Bases or Heights; (3) And thence, are also as their like Sides raised into the Figure's Dimension.

Suppose $L$, $l$, the Length or Sides of the Figures ($x, y$) or let $s, rs, s$, the Sides of the one, and $s, rs, s$, those of the other; then, if

If $B = b$, or $H = h$; $x : y :: BH : bb :: H : b$, or $:: B : b$.

All differ; $x : y :: BH : bb :: BL : bl :: HL : bl$.

For $B : b :: H : b$ (by 2. 63) :: $L : l$. (by 63) Therefore
Palmariorum Matheseos.

In \( \mathbb{S} \) Planes \( x : y : : (S^2 r : S^2 r : : ) S^2 : S^2 \).

2 Solids \( x : y : : (S^3 r : S^3 r : : ) S^3 : S^3 \).

And like inscrib'd Figures are as their respective wholes, Th. all similiar Surfaces, or solids, are as the Squares, or Cubes of their like Sides.

And if such Figures are equal, they have their Bases and Heights reciprocallly : : 1. And the contrary.

12. A Right Line bisecting an \( L \) of a \( \Delta \), cuts the op. Side directly :: 1 to the Legs. i.e. \( y : ? (:: A : B \text{ (by 11.63)} :: a \times e \times \) (by 26) :: a : e.

13. If from any \( L \) (BAE) of a \( \Delta \) Inscr. in a Circle, whose Diameter is \( AD \), a Perp. (AF) be let fall upon the op. Side, then (bec. \( \Delta \) ABF Sim. AED) \( AF : AB :: AE : AD \).

Also, let AC bisect \( L \), then (bec. \( \Delta \) ABC Sim. AEF) \( AC : AB :: AE : AF \), or BAE = CAF = AF^2 + CFA = AF^2 + BFE.

In any Plane \( \Delta \), whose Sides are \( a, b, c \), Height \( p \), and their op. Ls \( x, y, z \). Since the Diameter (of the Circumfer. Circle) \( d : a :: c : (ac \div -d) \) p. th. Area \( A = \frac{ab}{2} \), th. \( d : a :: cb : A \), or \( d : s, x :: cb : A \).

Again, \( \Delta \)s having one \( L \) common, are as the Fails of the Sides about that \( L \).

Also, since \( d : c :: (ab ; pb = 2A :: ) \frac{1}{2} ab :: \frac{1}{2} A \), or \( \frac{1}{2} d : c :: \frac{1}{2} ab : A \), th. \( \frac{1}{2} cd ; c^2 :: \frac{1}{2} ab : A \), that is, \( dxs, z :: c^2 :: s, x \times s, y : A \).

64. A Perpendicular demitted from the Right \( L \) of a Plane \( \Delta \) upon the op. Side, divides that \( \Delta \) into two, each similiar to the whole, and therefore to one another. For each has \( 1 \ L \), and \( 1 \ L \) common to the whole, therefore are similiar (by 63.) Whence

1. \( \sum a : x :: x : b \), \( \sum th. \ x^2 = ab \).
2. \( \sum e : y :: y : b \) by 63, \( \sum th. \ y^2 = eb \).
3. Th. \( x^2 + y^2 = ab + eb = a + e \times b = b^2 \).
4. Ina \( L \), \( b = \sqrt{x^2 + y^2} \), \( x = b^2 - y^2 \), \( y = b^2 - x^2 \).

4. And
4. And \( x^2 : y^2 (\therefore ab : eb) : : a : e. \)

5. \( x^2 : p^2 (\therefore ab : ae) : : b : e, \) & \( x^2 : p^2 (\therefore eb : ae) : : b : a \)

6. \( b + y : x : x : b + y; \) And \( b + x : y : y : b + x. \)

7. \( a : p : p : e : : x : y, \) th. \( p^2 = ae; ay = px; ex = py. \)

8. Th. To find a mean \( : : 1 \) between any 2 rights lines \( a, c. \)

Let \( a + e \) be made the Diameter of a Circle, the Perpendicular on the Point of meeting, and terminated by the Circumf. will be the mean reqd.

9. Whence we have the Method of making a Square \( \equiv \) to a given Pr. And th. \( \equiv \) to any Right-angled Figure given.

10. Also, To cut a Line \( (l) \) in Extrem and Mean \( :: 1; \) (i.e. that the \( \Box \) of the greater part \( (x) \) may be \( \equiv \) to the \( \Box \) made of the lesser part \( (1-x) \) and the whole Line.)

Make \( l \) and \( \frac{1}{2} l \) the Legs of a \( \perp \) \( \Delta; \) then the Hypotenuse \( \equiv \frac{1}{2} l \) sought. For \( x^2 = l^2 - lx, \) th. \( x^2 + lx = l^2; \)

th. \( x = l - \frac{1}{2} l^2 \equiv \frac{1}{2} l. \)

11. And, In any Plane \( \Delta, \) the Sides being \( b, x, y: \) Sup. lines \( (m, m) \) from the Vertex \( v, \) making each at the Base \( b \) a line \( L \) with that at \( v; \) let \( m \) be the part of the Base line intercepted between \( m \) and \( m, \) Then \( b^2 = x^2 + y^2 \pm xy, \) if \( v > 0, \) or \( < 90 \) degrees.

Then if \( v = 90^\circ, \) \( b^2 = x^2 + y^2; \) If \( v = 120^\circ \) or \( 60^\circ, \)

\( b^2 = x^2 + y^2 \pm xy. \)

Hence the Radius of any Circle is \( \equiv \) Chord of \( 60^\circ. \)

If \( v = 135^\circ, \) or \( 45^\circ, \) \( b^2 = x^2 + y^2 \pm xy \times \sqrt{2}. \)

If \( v = 150^\circ, \) or \( 30^\circ, \) \( b^2 = x^2 + y^2 \pm xy \times \sqrt{3}, \) &c.

12. In a Circle, the Circumscrib'd Square is to the Inscrrib'd as \( 2 \) to \( 1. \) For the (Diameter, or) Side of the Circumf., is the Diagonal of the Inscrib'd Square; but the Diagonal squared is to the Side squared, as \( 2 \) to \( 1. \)

Hence.
13. Hence the Diagonal (D) is Incommensurable to the Side (S) of the Square. For S : D :: \( \frac{1}{2} \), and (s) any part of S shall be in the same :: to \( \frac{1}{2} \), its corresponding part of D; and how small soever \( s \) be taken, yet \( \frac{1}{2} \) is capable of a further Division, and that in infinitum; therefore, 'tis impossible to find such a part, that will exactly measure D.

Whence the Infinite Divisibility of Quantity is evident.

14. If similar Figures (H, X, Y) be made on the Sides (h, x, y) of a \( \Delta \); that (H) on the Hypotenuse (h) is \( \approx \) the three (X, Y) on the Legs (x, y). For H, X, Y are as \( b^2, x^2, y^2 \), (by 11.63) But \( b^2=x^2+y^2 \), (by 2.64) th. H = \( \frac{X}{Y} + Y \).

Hence it is easy to find the Difference between any similar Figures; As also, the Sum of any Number of such Figures.

And the Quadrature of the Lunes of Hippocrates of Scio, is also evident. Let a, b, c be an Isosceles \( \Delta \), at a. Then
\[ m = \frac{1}{r} \] (by 4.64) th. \( m = x \).

15. Also similar Figures made on 4 : 1 Lines (a, b, c, d) are :: 1. As \( \text{sup. Sim. } \Delta \text{s A, B on a, b; and Sim. } \text{Paral. C, D on c, d; } \text{thence} \ A : B :: a^2 : b^2 \) (by 11.63)
\[ :: c^2 : d^2 \] (by Propor.) :: C : D (by 11.63)

16. In Obtuse, or Acute L'd \( \Delta \) a, \( c^2 = a^2 + b^2 + 2bx \).

For \( p^2 = ) a^2 - b^2 + 2bx + x^2 \)
\[ = c^2 - x^2 \]

17. Hence \( x = \frac{1}{b} \) \( a^2 + c^2 + b^2 - 2b \)

18. Also, given the 3 Sides of a Plane \( \Delta \), reqd. the Area.
(Fig. 2.) Since \( p^2 = c^2 - x^2 = a^2 + b^2 + 2bx - x^2 \), th.
\[ x = c^2 + b^2 - a^2 \]
\[ \& p = c^2 - x^2 \]
\[ \frac{1}{2} \]

\[ \text{or} \]
\[ \frac{1}{2} \]

\[ \text{or} \]
\[ \text{or} \]

19. A
19. A Tangent (PT) is a mean between the whole Secant (BT) and its extern. Part (CT).

For △TPC Sim. △BTP, therref. BT:

\[ PT : PT : CT, \text{ or } PT^2 = BTC. \]

20. Whence the \( s \)'s made of such Secants, from the same Point (T) and their outward Segments are ==. i.e. BTC == BTC. For each is == to the same □ (PT^2.)

21. Heref. those Secants are reciprocally as their outward Segments; i.e. BT : BT : CT.

22. Hence also, Two Tangents drawn from the same point are ==: i.e. PT == PT; For the Sq. of each is == to the same Rectangle.

23. And the parts of Crossing Chords are reciprocally : : 1: i.e. Bx : Cx : b : c, or Bxc = bC ; For △BxC Sim. △bxc.

65. Hence if \( t=\)Rad. \( d=\)Diameter, \( s=\)Sine, \( v=\)Versed-sine, \( c=\)Co-versed-sine, \( c=\)Chord, \( s=\)Co-sine, \( t=\)Tangent, \( s=\)Co-tangent, \( f=\)Secant, \( a=\)Arc.

1. \( dv = vv = s^2 - 2rv - v^2 \) for \( d-v : s : s : v \).

2. \( d : c : c : v \); for \( dv (= v^2 + s^2) = c^2 \).


4. \( d : C = c : C = c : V = v \), for \( C^2 = c^2 = dV = dv \)

5. \( st = r^2 \); for BT x DE = BCD

6. \( sf = r^2 \); for \( A_s x CB = ACD \)

7. \( sf = r^2 \); for \( C_s x CT = BCA \)

8. Th. \( t_A x t_A (= r^2) = s, a x t_A \)

9. And \( s, A x t_A (= r^2) = s, a x t_A \)

10. \( st = st \); for \( A_s x CB = C_s x BT \)

11. \( \sum rv = s, a x s, a x 2s, a \); for \( C_B = B_N x BA \)

12. \( \sum rv = s, a x t, a, a \) for \( C_B = A_s x B_x \)

13. And
13. And \( L,v = 2 L, s, \frac{1}{2} a - L, s, 30^\circ \)

14. \[ \frac{1}{3}r = s, \frac{3}{2} a; \quad \text{for} \quad \frac{1}{3} K \times CB = CN \times (\frac{1}{3} A K) \times CN \]
\[ \frac{1}{3} r = s, \frac{1}{2} a \times s, \frac{1}{2} a; \quad \text{for} \quad \frac{1}{3} K \times CD = \frac{1}{2} A \times D \]

15. \[ \frac{1}{2} x, \frac{1}{2} a = 2 s, d \times s, d; \quad \text{for} \quad CB \times A = AB \times CN \]
\[ \frac{1}{2} x, \frac{1}{2} a = 2 s, d; \quad \text{that is,} \quad CB \times B = AB \times BN \]

16. Also, \( s, a + r, t, a = t, 90^\circ - a \)

17. And it is evident, \( C \times (V \times H \times V \times E \times C \); \) But taking the Tangent \( \omega \) infinitely small, it will be its corresponding Arc, \( s \times e = r \times H \times V \); Conseq. the Sum of all the Sines erected on any Arc of \( \omega \) \( = \) \( r \). Th. The Sum of all the Sines erected on the Quadrantal Arc is \( = r^2 \).

18. Also, if \( e \omega \) be infinitely small, then \( C \times (C \times \phi \mu) \phi, \mu, \) and \( C \times (C \times \phi, \mu) \phi, \mu, \) \( = \phi, \mu \). Th. \( r = t, a \times t, a \).

19. \( C \times (C \times \phi, \mu) \phi, \mu, \) \( = \phi, \mu \times t, a \times t, a \).

20. \( s = s, t, a \times s, t, a \times s, t, a \times s, a \div r \)

21. \( s, a = 2 s, d \times s, d \times s, d \times s, d \times s, m \div r \)
And \( s, a = s, d \times s, d \times s, d \times s, m \div r \)

22. If \( A, B, C \), be Equidiff. Arcs, which call \( a, m \), their common Diff. \( d \), then \( C \times (C \times \gamma G) \gamma, L \).

23. Because \( s = s, d \times s, m \div 12, \) chord of \( A, a \), then op, produced to \( c, d \), will be \( s, \frac{1}{2} s + s \); Th. \( a \) \( (V - v) \).

24. Since \( r^2 = r^2 \), \( \frac{1}{2} (s, r^2 - s^2), \phi, \mu, \) \( \phi, \mu, \) \( a, \frac{1}{2} s, d \), \( \frac{1}{2} s, d \times s, d \times s, m \div 12 \), and \( s, \frac{1}{2} s + s, A \)

25. And
13. And \( L, v = 2 L_s \frac{1}{2} a - L_s, 30^\circ \)

14. \( \frac{1}{2} rv = s, \frac{1}{2} a \); for \( \frac{1}{2} K_e \times CB = CN (\frac{1}{2} AK) \times CN \)

15. \( \frac{1}{2} rv = s, a \times \frac{1}{2} a \); for \( \frac{1}{2} K_e \times CD = \frac{1}{2} A_g \times D \sigma \)

16. Also, \( s, a + r \); or \( t, a \); or \( t, 90^\circ - a \)

17. And 'tis evident, \( C_e \times (Ve) H = \pi = s \times s, \) for \( \Delta e V = \Delta e C \); but taking the Tangent \( s \), it will be its corresponding Arc, \( s \times s = r \times H \); Conseq. the Sum of all the Sines erect'd on any Arc (zY) is \( r \); Th. the Sum of all the Sines erect'd on the Quadrantal Arc is \( r^2 \).

18. Also, if \( s \) be infinitely small, then \( C' : C Y : : \phi \mu \)

19. \( C : C O : : a; \) \( P_l (O J) \); & \( C : C S : : M O : M P \)

20. \( \frac{1}{2} kq = 30^\circ \), and \( q x = q \), then \( yq = 60^\circ \). And \( \omega z = xy + \sqrt{4 q x^2} \) (or \( 4 \times \frac{1}{2} a x^2 = 4 x t^2 \) \( - x t^2 \) (\( q x^2 \))

21. Also \( u \) \( h = w + a \), \( e \); \( s, A \), \( A > 30^\circ \) \( s, A \) as much \( > 30^\circ + s, \) of its Defect \( \sqrt{3} . \)

22. If \( B_x, B_y, B_y \), be Equidiff. Arcs, which call \( a, m, e \); their common Diff. \( a \), then \( C_y : C R : : \gamma \) \( G (27 \beta) : : \gamma L \)

23. Because \( u = S - s \), \( m = \frac{1}{2} \) Chord of \( A - a \), then op, produc'd to \( d \), will be \( \frac{1}{2} S + s ; \) Th. \( (v - v) \)

24. Since \( r^2 - s^2 \) \( \frac{1}{2} \) \( (s) ; r \); \( s, (s) r^2 - s^2 \) \( - \frac{1}{2} r \times s \)

Th. \( s = \frac{s}{1} + \frac{1}{2} \frac{x^2}{3} \times \frac{1}{2} \frac{g x^2}{2} \beta + \frac{25 x^2}{6} \gamma + \epsilon \)

25. And
25. And \( s = \frac{a}{1} - \frac{a^3}{1 \times 2 \times 3} + \frac{a^5}{1 \times 2 \times 3 \times 4 \times 5} - \ldots \), &c.

26. Bec. \( \frac{dv}{v^2} = \left( \frac{a}{d} - \frac{v^3}{3} \right) r \div v \cdot \frac{d}{2} \times r \frac{v}{v} \)

Th. \( a = 1 + \frac{v}{6} + \frac{5 v^3}{40 d} + \frac{112 d^3}{152 d} + \ldots \), &c. \( \times \frac{dv}{v^2} \)

27. \( v = \frac{a^2}{1 \times 2 r} - \frac{a^4}{1 \times 2 \times 3 \times 4 r} + \frac{a^6}{1 \times 2 \times 3 \times 4 \times 5 r} + \ldots \), &c.

28. Th. \( s \left( r - \frac{v}{r} \right) = \frac{a^2}{1 \times 2 r} - \frac{a^4}{1 \times 2 \times 3 \times 4 r} - \ldots \), &c.

29. \( f \left( \frac{1}{s} \right) = 1 + \frac{l_2^2}{s^2} + \frac{l_4^2}{s^4} + \frac{l_6^2}{s^6} + \ldots \), &c.

30. Th. \( S \left( \frac{1}{s} \right) = \frac{a^3}{1} + \frac{a^5}{3} + \frac{a^7}{5} + \frac{a^9}{7} + \ldots \), &c.

31. \( s = \frac{t^2}{1 \times 2} + \frac{t^4}{3 \times 4} + \frac{t^6}{5 \times 6} + \ldots \), &c.

32. Bec. \( \frac{r^2}{s} = t, \quad \frac{r^2}{s} = a, \quad a \left( a^2 \right) = \frac{r^2 s}{s} = F, S, f \)'s.

33. And \( \left( \text{if} r = 1 \right) \quad \frac{a}{t} = \frac{1 + a}{t} \) (by 18, 65) =

\[
\frac{t \times 1 - t^2 + t^4 - t^6 + t^8 - t^{10} + \ldots}{t^2 + t^4 - t^6 + t^8 - t^{10} + \ldots}
\]

Th. \( a = t - \frac{1}{2} t^3 + \frac{1}{2} t^5 - \frac{1}{2} t^7 + \frac{1}{10} t^9 + \ldots \), &c.

34. And \( t = 1 + \frac{a}{2} + \frac{a^3}{4} + \frac{a^5}{6} + \frac{a^7}{8} + \frac{a^9}{10} + \frac{a^{11}}{12} + \ldots \), &c.

Th. \( f \left( \frac{1}{s} \right) = 1 + \frac{t}{2} - \frac{a}{3} \times \frac{a^3}{3} + \frac{a^5}{5} + \frac{a^7}{7} + \frac{a^9}{9} + \frac{a^{11}}{11} + \ldots \), &c.

35. \( S \left( \frac{1}{s} \right) = \frac{1}{2} a^2 + \frac{1}{2} a^4 + \frac{1}{2} a^6 + \frac{1}{2} a^8 + \frac{1}{2} a^{10} + \frac{1}{2} a^{12} + \ldots \), &c.

36. And if \( n = 2, \quad 305, 509, \ldots \) (to being Radius) then \( r = \frac{1}{2} a^2 + \frac{1}{2} a^4 + \frac{1}{2} a^6 + \frac{1}{2} a^{10} + \ldots \), &c. = \( L \), for \( L \), of \( a \), in \( Napper's \) Form; And \( 1 \) th thereof will give \( Briggs's \).

37. Given \( c \) the Chord of an Arc \( a \): Reqd. \( c \) that of another Arc \( A \), so that \( A : a :: n : 1 \). Since
Palmariorum Matheseos.

\[ a = \frac{e}{6d^2} + \frac{3e^2}{40d^4}, \text{ &c.} \]

\[ A = \frac{C^3}{6d^2} + \frac{3C^3}{40d^4}, \text{ &c.} \] (by 24)

Th. \[ C + \frac{C^3}{6d^2} + \text{ &c.} = n \times e + \frac{e^3}{6d^2} + \text{ &c.} = A \]

Th. \[ C = \frac{nc}{1} + \frac{1 - n^4}{2 \times 3d^2} \times e^3 - \frac{9 - n^2}{4 \times 5d^2} \times \beta + \frac{25 - n^2}{6 \times 7d^2} \times \gamma, \text{ &c.} \]

38. Bec. \[ t = \left( \frac{rs}{s} \right) \frac{rs}{\sqrt{r^2 - s^2}} \quad \text{if a be 30°} \]

\[ \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{2}} \]

And 64, or 64 \[ \frac{\sqrt{3}}{3} \times \frac{t}{3} - \frac{t^3}{3} + \frac{t^5}{5}, \text{ &c.} = \frac{3}{4} \text{ Periphery (π)} \]

But \[ 6 \times \frac{t}{\sqrt{3}} = \frac{\sqrt{36}}{\sqrt{3}} = \sqrt{12} = 2\sqrt{3}, \text{ and } t^2 = \frac{3}{4} \]

Let \[ \alpha = 2\sqrt{3}, \beta = \frac{1}{3}, \gamma = \frac{1}{3}, \delta = \frac{1}{3}, \text{ &c.} \]

Then \[ \alpha = \frac{1}{3} \beta + \frac{1}{3} \gamma - \frac{1}{3} \delta + \frac{1}{3} \epsilon, \text{ &c.} = \frac{1}{3} \pi, \text{ or} \]

\[ \alpha = \frac{3}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \frac{1}{9} \text{ &c.} \]

Thereof: the (Radius is to \( \frac{1}{3} \text{ Periphery, or } \frac{1}{3} \text{ Diameter} \) is to the Periphery, as \( \frac{1}{3} \), 1,000, &c. to \( \frac{1}{3} \), 14159265358979323 \( \times 4.62643383279508884197 \), 16939937510582097494 \( \times 4.592307816 \), 4062862689 \( \times 9862803482 \), \( \times 5342117057 \), 9999999999, True to above a 100 Places; as Computed by the Accurate and Ready Pen of the Truly Ingenious Mr. John Machin: Purely as an Instance of the Vast advantage Arithmetical Calculations receive from the Modern Analysis, in a Subject that has bin of so Engaging a Nature, as to have employ'd the Minds of the most Eminent Mathematicians, in all Ages, to the Consideration of it. For as the exact Proportion between the Diameter and the Circumference can never be express'd in Numbers; so the Improvements of those Enquirers the more plainly appear'd, by how much the more Easilie and Ready, they render'd the Way to find a Proportion the nearest possible: But the Method of Series (as improv'd by Mr. Newton, and Mr. Halley) performs this with great Facility, when compared with the Intricate and Prolix Ways of Arhimedes, Vicius, van Ceulen, Metius, Snellius, Lansbergius, &c. Tho' some of them were said to have (in this Case) set bounds to Human Improvements, and to have left
nothing for Posterity to boast of; But we see no reason why the indefatigable Labor of our Ancestors should restrain us to those Limits, which by means of the Modern Geometry, are made so easy to surpass.

Variety of other Series might be given for this purpose, tho' probably none so simple as this.

39. If \( r = 1,000, \) &c. the Length of the Arc of \( z \) will be \( \left( z, 1,159, \&c. \div 180 \times 60 \right), 00029,08882,08665,721 \\
59,61539,48461,\&c. \left( = \lambda \right) \); Let \( n \) be the Number of Minutes in any other Arc \( a \), then will \( a = n \lambda \): So that, by the foregoing Rules and Series, the Natural Sine, Cosine, Tangent, &c. of any Arc is readily made to any desired Number of Places; as also their Logarithms, and that from the Arc itself.

Other Theorems might have bin added for the Calculating of a Table of such Numbers; but these alone were thought sufficient to perform the fame, by various easy Ways.

66. Let \( BL, PB \), be any two Arcs \( (A, a) \) Then

\[ ML = s, A; \quad MR = s, a; \]

\[ LH = LR = s, A + s, a; \]

\[ IH = LR = s, A - s, a. \]

\[ Dn = v, A + a; \quad DK = v, A - a. \]

\[ dn = v, A + a; \quad dK = v, A - a. \]

\[ DL = s, A + a = 2s, \frac{1}{2} A + a; \]

\[ DL = c, A - a = 2s, \frac{1}{2} A - a. \]

\[ in = s, A + a; \quad LK = s, A - a. \]

\[ dH = s, A + s, a; \quad DR = s, a - s, A = v, A - v, a. \]

\[ \frac{1}{2} dl = s, \frac{1}{2} A + a; \quad \frac{1}{2} dL = s, \frac{1}{2} A - a. \]

Here 'tis plain, the Trapezia \( LDHK, LDKn, LDHa, LDKR \) are similar; For each has two \( \angle s \) at the Circum. (viz.

one \( = \frac{1}{2} A + a \), the other \( = \frac{1}{2} \) the Supplement of \( A + a \),) as also \( 2 \angle \Delta s \), form'd by the respective Diagonals and Sides, which are respectively similar; therfor, the Sides about the \( = \angle s \) will be :: 1, Conseq. the Trapezia are all similar.
By this Scheme, which I receiv'd from the Learned Mr. Halley, an endless Variety of useful Theorems may be deduc'd, and that by Inspection only: Such as these for Instance.

1. \( s \cdot A + a : s \cdot A - s \cdot a : s \cdot A + s \cdot a = a \)
   \( \vdots \)
   \( s \cdot A + a : s \cdot A = s \cdot a \)

2. \( s \cdot A + s \cdot a : s \cdot A - s \cdot a = \delta \cdot A + a : \delta \cdot A - a \)

3. \( v \cdot A - v \cdot a : s \cdot A + a = s \cdot A - a : \delta \cdot A \)

4. \( s \cdot A - a : s \cdot A + s \cdot a = s \cdot A - a : s \cdot A + s \cdot a \)

Thereof in Equidif: Arcs, (where \( m \) is the Mean between \( a \) & \( e \), its Equidiff. Extremes) tis, \( s \cdot m ; s \cdot e + s \cdot e : s \cdot m : s \cdot A + s \cdot E \) (as in Art. 23, 54.)

67. The Diameter of a Plane Figure is a Right Line bisecting all Parallels to the Tangents. The Lines bisected are called Ordinates to that Diameter. The Point of Concurrence of all Diameters is called the Centre of the Figure. The Intersection of the Curve and Diameter is called the Vertex.

That Diameter to which the Ordinates stand at right Angles, is called the Axis. The common Intersection of the Diameter and Ordinate is called the Point of Application. The Segment of the Diameter intercepted between the Vertex and the Point of Application is called the Abscissa or Intercepted Diameter.

And Geometric Lines are distinguished according to the Dimensions of the Equation expressing the Relation between the Ordinates and Abscissae; or according to the Number of Points in which they may be cut by a right Line. Then a Line of the First Order, will be a Right Line; Those of the Second, or Quadratic Order, will be Conic Sections; and those of the Third, or Cubic Order, will be the Cubic Paraboloid, &c. Therefore a Curve of the 1st, 2d, 3d, &c. Kind, is the same with a Line of the 2d, 3d, 4th, &c. Order.

And
And a Line of the Infinitemal Order, is that which a right Line may cut in an Infinite Numbers of Points; as the Spiral, Cycloid, &c. and all those generated by Infinite Revolutions of a Radius.

68. Of Curves of the First Kind.

If in any Right Line, upon a Plane, there be taken two Points $\varphi$, $f$; an Infinite Number of Points (p) may be found, so that $fp \pm \varphi p = vu$ or $fl$ a given Right Line, and $pp = pl$ everywhere. For with $vu$ (or $fl$) from $f$, describe the Arc $gl$, draw the Lines $f l$, make the $\angle lpp = \angle g l p$, or $pl = pp$; and the Curve drawn thro' $p$.

1. If it be $fp \pm \varphi p$, is called an Ellipse, or Hyperbola; as in Fig. 1. & 2.

2. If $f$ be infinitely distant from $\varphi$, the Arc $gl$ will become a Right Line perpendicular to $vu$, and the Curve is called a Parabola.

3. If $f$ and $\varphi$ coincide, the Curve will be a Circle, whose Radius is $\frac{1}{2} fp + \frac{1}{2} pp$.

Therefore, if upon a Plane, any Right Line $vu$ be taken, and therein $vp = uf$, and the Points in within, or without $\varphi$ and $f$; and Arcs described with $vm$, $um$, from $\varphi$, $f$, will intersect in the Curve of an Ellipse, or Hyperbola.

And in any indefinite Right Line $fg$ (Fig. 3.) let $vg = vp$, and $pa$ Perp. $fg$; with $sa$, from $\varphi$ out $pa$ in $p$, $x$, the Curve Line drawn thro' $x$ them will be a Parabola.

The Point $v$ is the Vertex, $pa (y)$ an Ordinate, $va(x)$ the Abscissa, $\varphi$, $f$ the Foci, $vu (z)$ the Transverse Axis.

'Tis evident the Parabola has but one Focus $\varphi$, so that $z$ is Infinite. Therefore the Properties of the Parabolic Curve are easily drawn from those of the other; yet to gratifie the Learner, we chose to do them separately.

In the Ellipse and Hyperbola.

1. Thro' $p$ draw $pt$ making $\angle lpt = \angle lpp$; Then shall $pt$ be a Tangent to Curve in $p$. For, making $pl = pp$, and taking any Point $e$ in the Line $pt$, then $ep = el$, but $fl (vu) < fe + al$ (in the Ellipse), and $fl (vu) + al > fe$ (in the Hyperbola). Th. $e$ is without the Curve.

2. Draw
Place this Figure at Page 246.
2. Draw ce \parallel \text{fp}, cutting ty in \(s\), and \(xp\) in \(o\). Then L
\(\text{me} (\text{lpq}) = \text{lope}. \) And \(\text{co} = \text{do} = (\text{bec.} \text{cc} = \frac{1}{2} \text{cf}, \text{and) sb.}
\text{co} = \frac{1}{2} \text{fp} \times \text{fp}, \text{Th.} \text{cs} \left(\frac{1}{2} \text{fp} \times \frac{1}{2} \text{fp}\right) = \text{cv}. \text{Th.} (\text{bec.} \phi \phi, \phi \text{sp}
\text{are in a) Semi-c. whose Centre is o) the L \text{me} = L.}
3. Let \(cv^2 (\frac{1}{2}t^2) : \text{nu} (s + q \times q) : : \text{vu} (r) : p = \text{Parame-
ter}; \text{Then (bec.} \frac{1}{2}t^2 (\frac{1}{2}t \times t) : \frac{1}{2} \text{tp} (\frac{1}{2}t \times p) = \frac{1}{2} \text{t}^2 : \frac{1}{2} \text{t}^2 \times q \times q, \text{or} \text{nu.}
4. Draw \(cg \parallel \text{tp}, cutting pf in \(v\), Then (bec. \text{cv} \parallel \text{pe}, \text{and}
\text{ce} \parallel \text{fp}) \text{pv} \text{(ca)} = \text{cv}, \text{by 2.) Supply the Hyperb. Figure.}
5. From \(p, \text{sbro} \phi \text{ describe a Circumf. cutting \(vu\) in \(s\)}
\text{and \(fp\) in \(d, l\); Bisect \(fd\) in \(v\); then}
\frac{1}{2} \text{fl} = \frac{1}{2} \text{fp} = \frac{1}{2} \text{r} = \frac{1}{2} \text{fd}, \text{et c. cv : cf : : ca : fr, th.}
1. \text{cv : cv} = \text{cf : ca : ca} = \text{cv : cv} = \text{cf} = \text{ca} + \text{fr}.
\text{[that is, cv : fr} = \text{ua : fp = fa, } \text{And}
2. \text{cv : cv} = \text{cf : ca} = \text{ca} \text{ofr : cv} \text{cv : ca} = \text{cv} = \text{fr} \text{+ ca} = \text{fp}
\text{[that is, cv : fr} = \text{ua : fp = fa, } \text{Th. cv}^2 : \text{rfu} = \text{vau} = (\text{fp}^2 - \text{fa}^2) \text{ap}^2.
6. \text{e.} \frac{1}{2} t^2 : t + q \times q : : t + x \times x : y^2 : : p : : t + X \times X \times Y^2
7. \text{And (bec.} \text{lpq} = \frac{1}{2} \text{lpq = pdq, sb. tp \parallel \text{pd \parallel cv,}) fr}
\text{fc} = \text{ca : ct. but. (by 5.) fr : fc} = \text{ca} = (\text{cv} \text{cs}) \text{ct,}
\text{that is,} \frac{1}{2} s = x : \frac{1}{2} t^2 : \frac{1}{2} t^2 \times a \text{ (which is also true, if} \(s
\text{be the Diameter}) : : (\frac{1}{2} + \frac{1}{2} + x \times x : \frac{1}{2} t + \frac{1}{2} t + x : s) : : x + a
\text{[\frac{1}{2} t + \frac{1}{2} t + x : s : : x + a : s) : : x + a}
9. \text{Or,} \frac{1}{2} t - x = x : \frac{1}{2} t = x + a : a. \text{Th. u} \times c = \text{or \: ap = vs}
10. s = a : \frac{1}{2} t = a : x + a = a. \text{Th. u} \times c = \text{ap = vs}
11. \text{Also,} s = p : (\frac{1}{2} t - x \times x) : \frac{1}{2} t + x = x + a : y^2
12. \text{Th.} \frac{y}{x} = \frac{1}{2} + \frac{1}{2} x \times x : \frac{1}{2} + \frac{1}{2} x \times x : \frac{1}{2} t = \frac{p}{x} \times \frac{p}{x}, \text{or} \text{ap} = \text{ca \times p}
13. \text{If} \text{vs, u} \times \text{(\parallel ap) meet ty in} s, \Sigma \text{; then}
\text{vs: ap \parallel (vt: at: \text{cv: u} \times (by 8.))} \text{Th.}
\text{vs \times u} \times \times \times \text{cv \parallel (ap \parallel (vt: at: \text{cv: u} \times (by 8.))} \text{th.}
\text{vs \times u} \times \times \times \text{cv \parallel (ap \parallel (vt: at: \text{cv: u} \times (by 8.))} \text{th.}
Th. vs x uς = \frac{1}{4} p x \frac{1}{v} = cτ x ap.

Note, in the Hyperbola, supply the Fig. because here omitted to avoid Confusion; And observe the same in other places where 'tis required.

15. If from the Intersect. of any Tangent to a Circle drawn to the Circle, and a Perpendicular to its Axis, then (bes. Δs tvv, tuς, Sim. Δς tvp, tuς) uς : nτ : : ap : vs.
Th. vs x uς = \frac{1}{nτ} x τv = (\frac{\text{vovfu}}{v} x \frac{\text{vfu}}{u}) \text{vovfu}
or \frac{v}{\text{v}} x \frac{\text{fu}}{\text{u}} = \frac{\text{vovfu}}{\text{u}} (\frac{1}{\text{ap}} = \text{cτ} x \text{ap}) Th. \phi and \tau are the Foci.

16. Let vs, uς (Perp. vu) intersect any Tangent to in \xi, \Sigma, draw \xi v, \Sigma; Then (bes. \text{vsxuvs} = \text{ vv} x \text{v} by 14.) Th. uς : \frac{\text{uf}}{\text{v}} : : \text{iv} : : \text{vs}, and the INCLUDED Ls \Sigma uς, \text{uf are} \angle, \text{th. άς are Sim. and L Σ uς} \angle \text{viv, but} \L \text{uf} + \text{uf Σ} = \angle \text{L uf} + \text{L ufΣ} = \angle \Sigma. \text{And} \text{ufu} = \text{uf} \text{ufΣ} \text{Th. if} \Sigma \text{be made the Diameter of a Circle, the Circumf will pass thro' the Foci} \phi, \tau. \text{(Supply the Figures.)}

17. Draw \xi \parallel \text{tp}, cutting \text{cp to} a; Then \text{za} = \text{an}.

For (producing \text{za to} \text{n, and} \text{cp to} \text{cp,} \text{ω}) \frac{\text{za}}{\text{in}} = \frac{\text{ap}}{\text{at}}.
\text{ca} \times \frac{p}{s} = \frac{dv}{\text{vω}} \times \frac{p}{s}, \text{th. cv x n x z} = (\text{vω x z})^2
\frac{\text{vω x viu}}{\text{v}} (by \delta, \text{th. cv x in x iz} \frac{v}{\text{ω}} x \frac{\text{ω}}{ω} x \frac{\text{ω}}{ω} = \text{cr}^2 = \frac{\text{vω x viu}^2}{\text{vω x cr}^2 or \text{iz} \frac{v}{\text{ω}} x \frac{\text{ω}}{ω} x \frac{\text{ω}}{ω} \pm \frac{\text{vω}^2}{\text{vω}} \text{cr} \text{vω}) \times \frac{v}{\text{ω}} x \frac{\text{ω}}{ω}
\text{iz, th. Δ cvv = Δ cvx} \frac{v}{\text{ω}} x \text{iz} = (by the like arguing) Δ\text{cmω} \frac{v}{\text{ω}} x \text{Δω} = \text{and Sim. D are} \text{th. za} = \text{ar}; \text{which therefore are} \text{Ordinates to the Diameter pq.}
Therefore all Lines passing thro' the Centre are Diameters; And all Diameters pass thro' the Centre; And all Ordinates are parallel to the Tangent as the Vertex of their Diameters.

18. Hence
18. Hence (bec. ve : ap (::vc : ac) :: tc : vc (by 6)
\[ \Delta \text{cpt} (\Delta \cap \text{tap}) = \Delta \text{cvw}, \text{st.} \Delta \text{tvs} = \Delta \text{spw} \]
And Fig. vixw = \( \Delta \) inz, also Fig. aepn = \( \Delta \) axz, i.e.
\[ \text{ap} \times \text{an} + \text{pt} \times \text{ap} = \text{ax} \times \text{az} = \text{ap} \times \text{sn} \times \text{an} + \text{pt} \times \text{ap} \]

19. And \( \text{pa} : \text{ps} (::ax : az) :: 2pt : P = \text{Parameter} \)
to the Diameter pq. For \( P \times \frac{\text{paq}}{\text{pq}} (\frac{ax \times 2pt}{ax} \times \frac{\text{paq}}{\text{pq}}) \equiv \frac{\text{az} \times \text{paq} \times \left(\frac{2pt}{pq}\right)}{\text{pc} \times \text{st}, or \frac{\text{tp} + \text{na} \times \text{(sn) ap}}{\text{or}} \right) \frac{\text{ax} \times \text{az}}{ax} \equiv \frac{\text{az} \times \text{ax} \times \text{a} \times \text{x}}{ax} = \frac{\text{az}^2}{ax} = \frac{ax^2}{ax}.

Therefor. the Squares of the Ordinates to any Diameter (D) are as the Rectangles under the Segments \( \text{of the Diameter, i.e.} \)
\[ D = x \times x :: y^2 (::D : p) :: D = x \times x \times x \times x \times x \times x \]
And (bec. \( \frac{1}{4}d^2 : \frac{1}{4}x^2 : d : p \)) of (the Conjug. Diam.) is = \( \sqrt{dp} \).

20. Let \( \phi = r, \nu \phi = q, \phi \phi = \delta \), then
If \( x = q, (x + q \times q) : y^2 \), then \( y = \frac{q}{p} \)
And \( t : 2y :: t + q \times q : y^2 \), then \( y = \frac{t + q}{2} \times \frac{q}{t} \)

21. Since \( \frac{1}{2}t^2 : (\frac{1}{2}t \times t \times q = \frac{1}{2}t^2 \times q) \times q = \frac{1}{2}t^2 \times q^3 \)
\( \frac{1}{2}t : \frac{1}{2}q \), then \( \frac{1}{2}t^2 : \frac{1}{2}t + \frac{1}{2}q \), i.e.
\( \frac{1}{2}t : r :: \frac{1}{2}t + \frac{1}{2}q \), then \( \frac{1}{2}t : r :: \frac{1}{2}t + \frac{1}{2}q \), Th.
\( r = \frac{1}{2}t + \frac{1}{2}q \), then \( q = \frac{1}{2}t^2 + \frac{1}{2}t \times \frac{1}{2}t + \frac{1}{2}q \), Th.

22. Sup. any Ordinate \( \phi r \), of an Ellipse, to meet the Focal Tangens in \( \gamma \); Then \( \phi r = my \). Let \( cp = r, cv = b, cm = n, \) then \( \phi r \left(\frac{vu}{cv}\right) = \frac{b^2 - r^2}{b} = \frac{1}{2}p, \) ct \( \left(\frac{cv^2}{c\phi}\right) = \frac{b^2}{r} \)

\( K k \)
\( \text{sb. pt} \)
250 Synoptes Part. 2.

\[ \text{th.} \phi t = \frac{b^2 - r^2}{r}, \text{ and mt} = \frac{b^2 + mr}{r}, \text{th.} (b: r:: \frac{b^2 + mr}{r}) \]

\[ \frac{b^2 + mr}{b} = m\gamma; \text{ But} \ (\bec. \ cv: \frac{c}{r} :: \nu m u :: mr^2, i. e. b:\]

\[ \frac{b^2 - r^2}{b} :: \frac{b^2 - n^2}{b}, \text{mr}^2 = \frac{b^4 - 2b^2r^2 + r^4 + r^2n^2}{b^2}, \]

and \( \phi m^2 = r^2 - 2rn + n^2, \text{th.} \phi r^2 = \frac{b^4 + r^2n^2 + 2rn^2}{b^2} \)

and \( \phi r = \frac{b^2 + mr}{b} = m\gamma; \) 'Tis the same in the Hyperbola

and Parabola.

23. Therfore, if cb meet the Focal Tangent in t, then ct (\( \phi b \)) = cv = cu.

24. (Bec. \( \frac{cv^2}{\phi} \) (ct) : cv (ct) :: \( \frac{cv}{\phi} \) + cv (\( \frac{ct}{\phi} \)) : cv

\[ \phi = ) u\Sigma = up, \text{ or} \ vs = v\phi. \]

25. The same being still supposed, (that is, that x = q) draw pu cutting any Ordinate mr in \( \theta \); Then \( \gamma \theta = \phi m. \)

( Supply the Figure.)

26. If on vu, a Circle be described, then bec.

\[ \text{ap}^2 :: \text{mr}^2 :: (:: \text{vau} :: \text{umv} ::) :: a\omega^2, m\phi^2, \text{th.} \]

ap \( :: \text{mr} :: a\omega :: m\phi :: \text{mr} :: r\phi, \)

Or ap \( :: a\omega :: (:: \text{mr} :: m\phi :: cb :: cv ::) :: e :: t. \)

Th. all (ap's: all a\omega's: ) Ellipse: \( \circ :: c :: t :: ct :: t^2 \)

\[ \circ :: \nu t :: (\circ \nu t^2) :: r. \text{ Th. Ellipse} = \circ \nu t. \text{ And the} \]

Ellipses \( E :: e :: TC :: tc :: (\bec. e = \nu tp) \)

\( \frac{p^2}{t^2}; \frac{c^2}{t^2}. \)

27. Parallelograms circumscribing an Ellipse, having their Sides parallel to the Conjugate Diameters, are equal. For since \( \Delta m\phi p = \Delta m\phi t \) (by 18) and \( \text{at} \mu., \text{Th. } \tau \mu \): \( \mu p :: \mu t :: b\mu, \text{th. } \tau p :: \mu p :: b\mu :: \mu t \) \( \Delta \tau p :: \Delta \mu p :: \Delta c\tau b :: \Delta c\mu b :: \Delta c\tau b, \text{but} \Delta \tau p = \Delta c\tau b \) (by 18) \( \text{th. } \Delta \mu p \Delta c\mu b, \text{and} \Delta c\mu \mu (\square \square) = \Delta \mu c, \text{th. } \Delta c\beta = \Delta \tau c, \text{th. } \square c\beta = \square ct; \text{ Also } \square \nu c (2\Delta c\nu y = 2\Delta c94 = 2\Delta cv\beta) = \square \beta c, \text{but} \)

\[ \square \nu c :: \square \nu c (:: c :: \tau p :: pt) :: \square \nu c :: \square ct \]

\[ \square \beta c :: \square \nu c (:: \beta \nu :: \nu k :: kb :: bl) :: \square \nu c :: \square ct. \]

Th.
28. If \( cp, c9 \) be semi-conjugate diameters, and \( p5 \) \( \perp c9 \), then \( c9 \times p5 (\equiv p9) = cv \times cb \).

Of the Hyperbolic Asymptotes.

29. Since \( cv = ca = cv^2 \) (by 6) if \( ct = o, \) then \( ca = \infty \).
    \[ vt = ut, \quad th. \quad vs = u \Sigma = vs \times u \Sigma |^2 = \frac{1}{4} tp |^2 \]; \( \theta \) if \( vE \) or \( vA \) (1 \( vU \)) be made \( = \frac{1}{2} tp |^2 \), then \( cE, cA \) are called asymptotes to the Hyperbola or Opposite Hyperbolas.

30. \( ca^2 = an^2 = (cv^2 = vE^2 = \frac{1}{2} t^2 = \frac{1}{4} tp = \tau = p; ...) \)
    \( vau = ap^2 = ca^2 - vau = an^2 - ap^2 \); but \( ca^2 > vau \), \( th. \)
    \( an^2 > ap^2 \); \( \theta \). \( n \) will always be without the Hyperbola, and conseq. \( cn, \theta \) infinitely continued, shall never meet the Curve.

31. Also \( cv = ca = vau, \) \( th. \) \( vE = vE (an^2 - ap^2) = npN \) or \( npN \); \( th. \) \( vE = tK or \frac{1}{2} K, \) \( th. \) \( npN = \frac{1}{2} K, \) but \( pN < \frac{1}{2} K, \) \( th. \) \( pn > 2t, \) \( th. \) the Hyperbola and its asymptotes do continually approach nearer.

32. And if \( cL, cK \) are asymptotes, and by any point \( (p) \) of the Curve, a tangent be drawn cutting \( cL, cK \) in \( 7, 5 \),
    \( th. \ p5 = \frac{1}{2} p \times Lr |^2 = p7; \) \( Lr \times 4 = 7p = 7p |^2 = \frac{1}{2} p \times 4p; \) \( th. \ Lr = 24 \).

33. Draw \( r3, \) parallel to any semi-diameter \( pc, \) cutting the asymptotes in \( B, \chi; \) \( th. \) \( B r \chi, \sigma \chi = cp^2 = \gamma \chi \). For the ordinates applied to the points \( p, \sigma, \) meet the asymptotes in \( 7, 5, \) and \( Lr, 4, \) \( th. \)
    \[ \frac{7p}{pc} \times \frac{5p}{pc} = \frac{Lr}{Br} \times \frac{r4}{\chi \gamma} \]; \( th. \) \( &c. \)
    \( \chi \sigma = B \).

34. If thro' any point \( P, \sigma, \) of the Hyperbola, be drawn right lines \( P3, \sigma 2, \) parallel to the asymptotes, \( th. \) (bec. \( \Delta 2u = \sigma \), Sim. \( \Delta P3, \) and \( \sigma \mu = P3, \) \( th. \) \( 2 = y 3, \) and \( \mu = P3, \)) \( P3 ; \sigma 2 = (\mu 2 = y 3 ; \mu c = c y ; c 3 = P3 ; c y - \sigma 2 = c 2 ; c 3, \) \( th. \) \( P3 \times 3c = \sigma 2 \times 2c. \)

35. If the segments \( c 4, c 9, c 3, \) of the asymptote \( cy \) be \( \infty \), then shall \( \sigma 9, \sigma 4, 3p \) (drawn \( \parallel \) to the other asymptote \( c 4 \)) be \( \infty. \)
For \( yv : iS : ( c' : c' : c' : c' : ) iS : 3P. \)

But if \( c' \neq e' \oplus d' \oplus f' \), &c. that is

if \( c' \), \( e' \), \( d' \), \( k' \), &c. be in a Contin. Arith. Proport.

Then \( yv \), \( e' \), \( iS \), \( k' \), &c. are in a Contin. Harmon. &c. as his case to demonstrate.

Let \( yS \parallel y\mu \), then \( iS : yS : P^3 : yP \), and \( d' : yS : yv : yv \), th. \( d' : yS : P^3 \times yv : yP \times yv \), but \( d' = P^3 \times yv \) (by 35) th. \( yS = yv \) or \( y\mu \), th. \( yS \) is a Tangent, and \( P^3 \) an Ordinate to the Diameter \( CS \), th. \( CS = \Delta \mu S \), and Space \( S^3 = S^3 \), but \( \Delta cS = \Delta cS \), th. Space \( S^3 = 3^3 \), but \( \Delta cS = \Delta cS \), th. \( S^3 = 3^3 \), but \( \Delta cS = \Delta cS \).

Likewise, if \( c' \), \( d' \), \( e' \), \( k' \), &c. draw \( k' \), \( e' \), \( d' \), \( e' \), th. \( S^3 = S^3 \times P^3 \), \( e' = P^3 = k' \), \( S^3 = S^3 \), \( P = P \), \( e' = \mu \), and the Logarithms. So the Space \( \mu P \) is the Log. of the Ratio \( P^3 = \frac{cS}{c^3} \); And the Log. of the Ratio of Equality \( \frac{\mu P}{\mu} \) is 0.

In the Parabola.

36. Let \( yS = q \), then \( Sg (SP) = x + q \), \( Sg = x - q \)

and \( y^2 (PP^3 - 2q = x + q \) \)

And \( 2q \) is called the Parameter of the Axe.

37. Th. \( yS = \frac{2q}{y} \); And \( 2q = 2y = \frac{y}{z} \).

38. Also \( y^2 : x^2 \) \( ( : px : pX ) : : x : X \).

39. Since \( x < X \), th. \( y < X \), th. the Curve of a Parabola runs off infinitely.

40. \( px = x \) \( (PX - px) = x + y \) \( (y^2 - y^2) \) \( = y \) \( (x + y) \) \( \) \( that \ is \, px = x \)

41. \( \frac{y^2}{(x + x) : y : y \times e} \)

Th. \( \frac{y^2}{(x + x) = y} \)

42. Any
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42. Any right Line nz cutting the Axe and Curve, will again if produced, cut the Curve in r, so that vr, vn, vm \( \frac{r^2}{ap^2} \); Suppose it so, let rm, zn \( \perp \) nm, then vr : vn : : vn : vm :: vr : vn + vm :: nZ : nm, and \( r^2 : mr^2 :: nZ : nm^2 :: vr^2 : vn^2 :: vn : vm \), th. the Point r is in the Curve.

43. Draw pt, so that tv = va, then pt shall touch the Curve in p; since any other point (s) in pt will be without the Curve: For (drawing us \( || \) pa) \( \frac{r^2}{ap^2} = \left( \frac{st^2}{at^2} = \frac{1}{2} \right) \)

\[ \frac{iv}{va^2} = \left( \frac{iv}{va^2} = \frac{iv}{va} = \right) \frac{r^2}{ap^2} \] Th. \( it > rz \).

44. Th. \( op (ga) = tp \); and if dp \( || \) fv, then \( Ldpv \) \( \left( \varphi \varepsilon \right) \) = \( \varphi \varepsilon pt \).

45. If pr \( \perp \) pt, then as \( = \frac{1}{2} r \); for \( (pxav) \frac{1}{2} px 2av = \left( \frac{ap^2}{2} \right) \) av \( \times \) as.

Th. if \( sx, \) so \( \perp \) pd, pp, then, \( (bec. \Delta spx, spo, spa \) are \( = \) and Sim.) \( px = po = \frac{1}{2} a = \frac{1}{2} p \).

46. And since \( op = \varphi t \), and \( L \varepsilon pt = \left( \right) \), the Points \( \varepsilon, p, t \), are in a Semi-Circ. whose Centre is \( \varphi \); Th. \( op = \varphi s = \varphi t \).

Hence the \( L \) made by the Tangents, from the Extremities of any Ordinate, is \( \frac{1}{2} \) that at the Focus made by Lines from the same Extremities.

Th. if the Ordinate passes thro' the Focus, the \( L \) made by the Intersection of its Tangents will be a Right one.

And \( (bec. av = vt) vs^2 \left( \frac{1}{2} ap^2 \right) = \frac{1}{2} av \times p \).

47. Let any Rt. Line zn, drawn \( \parallel \) tp, cut the Curve in \( z, r, \) and lpd \( \left( \parallel \right) va \) in \( a \); let rm, zn \( \left( \parallel \right) pa \) cut pa in n, th. Then ra = za.

For \( \Delta apt \left( \mathcal{O} wa, bec. \varphi a v = \varphi t \right) \Delta zn \left( \varphi ap^2 = rz^2 \right) ; \varphi av : iv \left( \mathcal{O} wa : \mathcal{O} \omega, \right), \th. \Delta zn = \mathcal{O} a \); and (by the like arguing) \( \Delta mn = \mathcal{O} m, \th. Fig. \varphi zm \left( \varphi mn \right) = \mathcal{O} \omega m, \th. \Delta ran = \) and Sim. \( \Delta za \; \mathcal{H} b, \th. \Delta ra \; \mathcal{O} za, \th. \) and pd is a Diameter, whose Ordinate is rz.

Hence, in the Parabola, every Line parallel to the Axe is a Diameter. And all Diameters are parallel to the one to the other. Also all Ordinates are parallel to the Tangent at the Vertex of their Diameter. Therefor, a Line, thro' the Vertex of the Diameter, drawn parallel to the Ordinate is a Tangent to the Parabola.

48. And
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48. And Fig. \( \triangle \text{mp} \) \( \Delta \text{ra} \) \( \Delta \text{pq} \) \( \Delta \text{st} \). 
\( \Delta \text{mn} \) \( \Delta \text{at} \). 
\( \text{ib.} \). \( \Delta \text{an} \). 
\( \text{i.e.} \). \( 2 \text{pt} \times \text{pa} = \text{at} \times \text{an} \). 

49. Let \( \text{pa} : \text{ps} : : 2 \text{pt} : \text{P} \); \( \text{i.e.} \). \( \text{va} : \text{pt} : = \text{pt} : \text{P} = \text{Parameter to the Diameter} \text{pd} \); For \( (2 \text{pt} \times \text{pa} \times \text{at} \times \text{an} \text{or } \text{ps} \text{or } \text{pt} \text{or } \text{P}) \)

\[ \frac{\text{P}}{\text{pt}} = \text{pa} \times \text{P} = \text{at} \times \text{an} \times \text{at} \text{ or } \text{ps} \text{ or } \text{pt} \text{ or } \text{P} \]

\[ \text{the Abscissa of every Diameter are as the Squares of their respective Ordinates.} \]

50. Since \( (\text{pt}^2 = \text{v} \beta^2) = 4x^2 + y^2 = \text{Px} \), \( \text{Th.} \)
\[ \text{P} = \frac{(4x + \frac{v^2}{x})}{x} = 4x + p \text{ (4q And P-p=4x).} \]

Hence \( p \) is the least of all the Parameters.

Also \( \frac{\text{P}}{x+q} = 2 \text{ or } \text{q.} \). And if \( \text{ds be} \ \perp \text{pt, then q} \text{ is } \text{ps} \)
\[ = (2 \text{ps} - 4 \text{pt}^2 \times \text{ps}^3) = x+q - x^2 - \text{q} = x+q \times \text{q} = \]
\[ \text{p} \times \text{q} \text{ ; Th. q}^2 \text{ is ever as q}. \]

51. If an Ordinate db to the Diameter dp passes thro' the Focus q, then \( \frac{\text{P}}{x+q} \times \text{v} \text{ or } \text{v} \times \text{v} \text{ by q} = \text{dp. Also (bec.} \)
\[ \text{p} \times \text{dp} = \frac{1}{4} \times \text{p}^2 = \text{db}^2 \text{ } \frac{1}{2} \text{P} = \text{db} = 2 \text{dp (dp + ps) is ts.} \]

These are the chief Properties of Curves of the First Kind, tho' there may be an innumerable Variety of others, most of which, if thought worth the Labour, may be drawn from the foregoing Principles. And such whose Curiosity prompts them to a farther Inquiry into these Matters, may have Recourse to larger Volumes.

69. Of the Conic Sections.

1. If a Cone be cut by a Plane thro' the Axe, the Section will be a Triangle. But if cut Parallel, or Subcontrary to the Base, the Section will be a Circle: For a Cone may be considered as composed of an infinite Number of Circles, all parallel to the Base; And the Subcontrary Section is evidently Similar to the Base, Bec.

\[ \text{Sim. } \Delta \text{pcv, } \text{th. } \text{pcxcv} = (\text{pcxcv}) = \text{ca}^2, \text{th. yap is a Circle.} \]

2. If in any Section of a Cone, a right Line (VP) terminating in the Curve intersect any two Parallel Lines (AE, DE)
Let the right Line VP be the Common Section of the
ΔNTR, (passing thro' the Vertex of the Cone, but not
thro' the Axe) and the Section AVE.

Then \( RC \times CN = EC \times CA \), and \( PC \times vC = EC \times ca \)
(by 64. 23.) Also \( \frac{RC}{EC} : CP : CP \), and \( \frac{NC}{VC} : \frac{Nc}{VC} \).

Th. \( PC \times vC = EC \times CN \cdot \frac{PC}{EC} = \frac{PC \times vC}{EC \times CN} \).

Or \( ec \times ca : EC \times CA = PC \times vC : PC \times CV \).

3. Th. In any Conic Section, if two
Parallels are cut by two others, and all
terminate at the Curve, the Rectangles of the Segments shall be Proportio-
nal, \( : ec \times ca : EC \times CA = PC \times vC : PC \times CV \).

4. Let any Plane AVE cut the Base
of a Cone in A, E, then NR drawn Perp. to AE, and
thro' the Centre, bisects AE. Let a Plane NTR cut the
Cone thro' NR and the Axe, and a Plane (nare) cut it \( || 
\) to the Base, then the com. Sect. \( ae \) and \( nr \), of the Planes
\( \overline{nr} \), \( \overline{ave} \), and \( \overline{nTr} \) are \( || 
\) to EA and NR, but \( nr \) passes thro' the Centre, th.'bisects ea in c ; also \( vC \) (the
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com. Sec. of the Plane AvE and that per Axem) bisects all right Lines drawn || to AE in the Section, th. vc is a Diameter of the Section, and AC, ac, are Ordinates. If the Plane per Axem be Perpendicular, or Oblique to the Base, vc will be the Axe, or Diameter of the Section.

5. Let NRT be a Δ thro' the Axe of a Cone, then all right Lines in the Plane of that Δ, from any Point v in the Side thereof, are Diameters of Curves of the First Kind.

1. And if a Plane thro' any of those Diameters, meet both Sides of the Δ per Axem below the Vertex of the Cone, the Section will be an Ellipse; But if it meet one (produced) above the Vertex, the Section will be an Hyperbola, and the Sections of the opposite Cones will be equal Hyperbolas. For


Th. vcu : vCu : : (ncr : NCR : : ) ac : AC²

2. Draw TM, vh || vu, NR ; make vu : vh : : us : P


The same appears more general by 69. 2. For (if D, d, be the Tranverses and Conjugate Diameters ; y, t, Ordinates to the Diameter D; x, X, their Abscissa) xx.D±x : X.x

D±x : : y² : x². And if an Ordinate (y) be applied to the Centre, 'twill become = ¼, then ¼D² : ¼d² : : xx.D±x : : y². Also, let (1) D : d : : d : P, And (2) D : x : : P : t, then (bec. D±x : : P±t : : t (by 2) th. xx.D±x


(by )
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(by 69 a) : x : t (by 2) : x x D + x : x x D - x : x x P - x ;

\[ y^2 = \frac{DPx - Px^2}{D}. \]
And bec. \( DP = d^2 \), \( dp = D^2 \),
\[ th. Pp = Dd, \] and \[ \frac{1}{4} d^2 \left( \frac{1}{2} DP \right) \equiv \frac{1}{4} Dx \cdot \frac{1}{2} P. \]

2. If the Plane be Parallel to one Side of the \( \triangle \) per Axem, the Section will be a Parabola. For

1. \( vc : vC : : \left( cr : CR : : ncr : NCR : : \right) ca^2 : CA^2 \).

\[ th. CA^2 (NCR) : TVC : : \left( NR^2 : NTR : : P : TV : : \right) \]
\[ PxvC : TVC, \] th. \( CA^2 = PxvC \). Or by 69, 2. (bec.
\[ D - x = D - X, \] since \( D = \infty \)) \( x : X : : y^2 : Y^2 \).

And if \( x : y : y : P \), then \( Y^2 = PX \) every where; For
\[ Px (y^2) : x^2 : \left( x : x : x \right) Px : PX. \]

6. If the Circle he describe, whose Segment vth
contains an \( L = L \) tvh, the Chords vth shall be the Parameters to the Diameters tv, \( vc, \) &c. For in the Ellipse and Hyperbola, (bec. \( \Delta vhz \) Sim. \( \Delta vsu \) th.) \( vuxP (vhxsu) = vuxvz, \) th. \( P = vz \). And in the Parabola (bec. \( \Delta vhz \) Sim. \( \Delta vRC \)) \( vC \times P (vh (NC) \times CR) \equiv vC \times vz, \) th.
\( P = vz \).

And ut Tangents to the Circle from the Extremities of the Diameters, shall be the Distance of the Foci; For ut^2 = vu^2 - vxvux (by 64, 19) = square of the Distance of the Foci (by 68, 22.)

7. In any Conic Section vpx, where \( ap, ax \) are Ordinates, \( v \) the Diameter, \( vP \) a Tangent; Let \( pc, vx (v \parallel va) \)
be infinitely small. Then (since \( ap^2 = p \times t \times x \times x va \equiv v2^2 \)
\[ = (bec. va : t : : 1 : \infty )p \times t \times x \times va, \] and \( so\)
\[ \alpha x^2 = \int p \times t \times x \times va, \] th. \( ap^2 = \alpha x^2 : : va : va, \)
\[ \text{Ap : ep : : t : : 1 : \infty .} \] And (bec. \( vP^2 = (vC^2 \]
\[ \text{+ cp}^2 \equiv (vC^2) vP = vC. \] Tb. Chord \( vP = \text{Tangents} vC = Arc vP; \) And \( pc : vx : : vP^2 : vP^2. \)

8. Let any Line \( vF \) cut \( pc, \) \( px \), produced, in e, f, then
\( cp : ce : : 1 : \infty , \) th. ec = ep, and \( \Delta vce = \Delta vfi. \) Tb. \( \Delta vpe : \Delta vFf : : (\Delta vce : \Delta vFf : : vC^2 \equiv vF^2) vCe^2 : vFi^2. \)

70. Curve Lines are by some distinguishing'd thus; That, whose Nature is express'd by an Equation, wherein the L

\[ \text{two} \]
two Indeterminate Quantities represent Right Lines; or one of them a Curve Line, they call an Algebraic, or Transcendens Curve: And those, the Exponent of whose Equation are variable Quantities, they call Exponential Curves.

But since the Relations between the Indeterminate Quantities in the Equation of a Curve may be infinitely varied, there, the Number of such Curves is infinite; Some of which, besides those of the First and Second Kind, Geometers have consider'd; But of these, we shall in this place, only take Notice of the Primary Properties of the most Simple Transcendents, and Exponential Curves; the Cycloid, and Logarithmic Line; their Use being very considerable.

71. If the Tangents to the Points of any Curve be produced till they become equal to their respective Parts of the Curve, their Extremities will be in another Curve; which is said to be describ'd by Evolving the former, the Curve Evolv'd is called the Evolutes, and the Tangents so prolong'd are called the Radii of the Evolutes, or Radii of the Curvature; Which therefore are all perpendicular to the Curve describ'd by the Evolution: So that the Points of the Evolutes are but the Intersections of Perpendiculars, to the Curve describ'd, that are infinitely near the one to the other.

Th. Sup. the Ordinates AP, ap, infinitely near, and C the Intersection of the Perpendiculars to the Curve VPP, in P and p. Draw CB || AV,

then Pe : Pp :: PB : PC, i : e  ⋅  

\[
\frac{x^2 + y^2}{z^3} : z : \frac{x^2 + y^2}{z} \times z : x,
\]

But whilst \(x\) and \(y\) increase by \(\frac{dx}{x}\) and \(\frac{dy}{y}\), PC becomes pC, th. does not increase. Conseq. the Flux. of \((PC)\)

\[
\frac{x^2 + y^2}{z^3} \times \frac{1}{z} = \frac{z \frac{dx}{x} + z \frac{dy}{y} + z \frac{dy}{y}}{x} = 0, \text{ th. (bec.}
\]

\[
\frac{\frac{dx}{x} + \frac{dy}{y}}{z} = \frac{\frac{dx}{x}}{z}, \text{ but Pe : Pp :: PB : PC, i : e \cdot x :}
\]

\[
\frac{x^2 + y^2}{z^3}.
\]
Evolvus or Radius of the Concavity of any Curve, or the Radius of a Circle that touches the Curve in $P$ or $p$.

72. The Curve Line describ'd by the Motion of any Point of the Circumf. of a Circle revolving on a Right, or Circular Base is called a Cycloid. Thus (when the Base is a Right Line) the Arcs $EP$, $BQV$, are those of the generating Circle, in their due places, whilst the Points $P$, $V$ of the Cycloid are describ'd: So that (drawing $PQ || CB$) $CE = \text{arc } EP = \text{arc } BQ$, th. $\text{arc } QV$ $(BE) = QP$.

Whence, to describe this Curve, make $d : : c : : BV : BC = \text{Arc } BQV$, divide $BC$, and the arc $BQV$ into the like Number of parts, in the points $E, e, Q$, and $Q, q, Q$. $\therefore$ draw $BQ$, $Bq$, compleat the Pgrs. $BQPE, Bqpe$, the Points $P$, $\pi$, will be in the Curve.

1. And where a part of some Curve $VQ = x$, and $PQ$

$= y$, then the Tangent $Qt = y \cdot \frac{x}{y}$. But if $x = \text{Arc}$ of a Circle, and $x : y = m : n$, th. $x = ym \div n$, th. $Qt = ym \div n = x$. But in the Primary, Protralkd, or Contr. Cycloid, $n = >, m$, Th. in the Primary Cycloid $Qt = y = x$. Th. $\ell tPQ (\ell tQA) = LVQA$, Th. the Tangens $Pt$ to any Point $P$ of the Cycloid is $||$ to the respective Chord $VQ$.

2. Th. $QA : AV : : PA : AT$; And $PT : TA : : QV$ $(\ell d \times \ell v) : VA (\ell v \times \ell v) : \ell d : \ell v$.

3. And
3. And (bec. \( VQ = vdv \)) and \( || PT \) \( v : vdv : : v \) \( d^3v \) \( \frac{1}{2} v = \frac{1}{2} \), th. the Cycloidal Arc \( VP = (2v dv) 2VQ \). i.e. The Curve is \( = \) twice the corresponding Chord.

4. If the Semicycloid \( CLd \) be Evolv'd, the Curve described will be a Semicycloid, Similar and Equal to the Evolute. For the Ray of the Curvature \( PL \) is a Tangent to the Cycloid in \( L \), th. \( \frac{1}{2} LP = \frac{1}{2} \) arc \( LC = Cy = LB \), th. \( EP = Cy (= BQ) \) but \( EP || Cy \), th. \( (VB = and || CD) \) a Circle thro' \( B, Q, V \), is \( = \) Circle \( DyC \), and the arc \( BQ = \) arc \( Cy = \gamma L = CE \). Th. the Curve \( CP \) is a Cycloid, equal and like to \( CLd \).

Th. if \( CDdM \) be two like Cycloidal Plates, a Weight suspended at \( d \) by the Chord \( dLP \), when put in Motion, will describe a Cycloid, whose Axis is \( \frac{1}{2} \) the Length of the describing Pendulum.

These things will also appear, by finding the Value of the Ray of the Evolute, according to the general Rule (Art. 71).

Since \( y = a + s \), th. \( \dot{y} = \frac{\dot{r} v}{s + r \dot{v} - v \dot{v}} \)

\[ s = 2r \dot{v} - v \dot{v} = \frac{v \times 2v - v}{v^2}, \text{th.} \dot{y} = - \]

\[ \frac{v^2}{v \dot{v}} \text{ vs, th. PL \( (or \frac{\dot{r} \dot{v} + \dot{y}}{v \dot{v}} \)} = \frac{8r^3 \dot{v}^6 + v^3 \dot{y}^2}{-v \dot{y}^2} = \frac{8r^3 \dot{v}^6 + v^3 \dot{y}^2}{-v \dot{y}^2} = 16r^2 - 8rv_\frac{1}{2} = 2 \times 4r^2 - 2rv_\frac{1}{2} \]

\[ = 2 \times AQ^2 + AB^2_\frac{1}{2} = 2QB = 2PE. \]

Th. if \( v = 0 \), \( PL = Vd = (2x4r^2_\frac{1}{2}) 4r = 2d \). And since \( LP \) is Perp. to the Curve and \( || BQ \) a Line \( PT \) \( VQ \) shall be a Tangent. Also any part \( (CL) \) of the Cycloidal Arc, \( = = \) to twice \( (Cy) \) the corresponding Chord.

The Cycloidal Space \( (VCB) \) between the Curve and the Circle is \( = \) (bec. its Elements are \( = \) to their corresponding Arcs, which are an Arith. Progr.) \( \frac{1}{2} \) \( \Box CBV = \frac{1}{2} \) or \( \approx \) Generating Circle.

73. Let two right Lines move at right \( \angle \), the one thro' equal Spaces in equal Times, the other thro' Spaces decrease.
decreasing in a Geometric Progression in the same Times, the Curve describ’d by their Interception is called the Logarithmic Line. Th. its Ordinates to the Equal Divisions in the Axe are Geometrically Proportional. This Curve therefore may be describ’d thus: Divide BP continually, so that BP : BR :: n : m, by making Bb (as bB = BC, &c.) = \frac{1}{n} BM, and drawing bp, b\pi, &c. || BP, also MP, MR, &c. then Rp, r\pi, &c. drawn || BM will give the Points p, r, &c. in the Curve. By the Generation of this Line, its plain.

1. If the Ordinates BP, bp, &c. are as Absolute Numbers, then Bb, Bg, &c. increasing, or BM, bM, &c. decreasing, as the Absolute Numbers Decrease, are as their Logarithms, according to Naper’s, or Briggs’s Method.

2. Bec. MP = Secant of 45° to the Radius BM and Tangent BP, Th. the Logarithmic Line is composed of all the Secants.

3. The Line BC continued, will be an Asymptote to the Curve PAC; And the Subtangent (s) or CT ( = y \times \frac{x}{y} ) is everywhere the same; For Y : Y’ :: y : y, and x (CA) is constant.

4. Since x = sy, th. Flux. of the Area (xy) = sy, Th. the Infinite Space AECc = sy = \sum ACT = 2\Delta ACT. Th. the Space between any two Ordinates Y, y, is = s \times Y - y.

74. If whilst a Ray, with an Equable Motion, describes the Circumf. of a Circle, a Point from the Extremity thereof, moving towards the Centre with a Velocity decreasing in a Geometric Progression, will generate a Curve called the Logarithmic Spiral.

1. The Arcs Ar, r\pi, &c. or the \ell s ACa, aCa, &c. being supposed infinitely small, and in an Arith. Progression; and the Radii, or Sides about those \ell s in a Geometric Progression, Th. the \ell s made by the Radii and Curve are every where equal.

2. If
2. If the Radii CA, Ca, &c. are as Numbers, the Arcs Ar, Aζ, &c. will be as their Logarithms.

3. If Ca = γ, eA = y, aA = z, tα = s, then, because \( \frac{z}{s} : : y : : \frac{m}{n} \), th. \( mz = ns \), th. \( s = \frac{z}{y}, \) i.e., The Infinise Spiral aC is = to the Tangent at.

75. The Ordinates in all Curves respect either the Axis, or a determined Points. And where the Ordinates are made parallel to one another, and right to the Axis; Let \( T = \) Tangent, \( τ = \) Subtangent, \( P = \) Perpendicular, \( ω = \) Subperpendicular, \( τ = x + \frac{ω}{x} \), and \( a = τ - x \). Then,

1. \( \frac{y^2 + x^2}{\frac{1}{2} + \frac{y}{x}} \) \( T = \frac{y^2 + x^2}{\frac{1}{2} + \frac{y}{x}} \) \( \frac{x}{y} \) \( \frac{y}{x} \).
2. \( \frac{y^2 + x^2}{\frac{1}{2} + \frac{y}{x}} \) \( \frac{y}{x} \) \( \frac{x}{y} \).
3. \( \frac{y^2 + x^2}{\frac{1}{2} + \frac{y}{x}} \) \( \frac{y}{x} \) \( \frac{x}{y} \).
4. \( \frac{y^2 + x^2}{\frac{1}{2} + \frac{y}{x}} \) \( \frac{y}{x} \) \( \frac{x}{y} \).
5. \( \frac{y^2 + x^2}{\frac{1}{2} + \frac{y}{x}} \) \( \frac{y}{x} \) \( \frac{x}{y} \).

6. (Bec. \( x + a = y \frac{x}{y} \)) \( a = y \frac{x}{y} - x \frac{y}{x} \).

The Values of any of these being made, from the given Conditions of a Problem proposed, respectively equal to their Values here, the Nature of the Curve will be found; and the contrary. Thus,

If \( τ = ny = y \frac{x}{x} \), then \( ny = \frac{y}{x} = x \), th. \( y = x \).

And if \( y = x \), then \( τ (y \frac{x}{y} = n) = ny = nx \).

Or if \( px = y \), then \( px = 2yy, \) th. \( \omega = (y \frac{x}{x} = y \frac{y}{x} \frac{x}{y} = 2yy) \).

76. To find the Length (λ), or Area (ω), of any Curve Line, or Curvilinear Plane.

1. Where
1. Where the Ordinates are parallel; \( \lambda = x^2 + y^2 \frac{1}{x} \), and \( \dot{\lambda} = \dot{x}y \).

2. Where the Ordinates respect a Point; (Let \( r = \text{Rad. of a Circle describ'd on that Point, } x = \text{Abscissa, and } y = \text{that part intercepted between the Point and Curve} \) \( \lambda = y^2 x^2 + r^2 y^2 \frac{1}{r} \), and \( \dot{\lambda} = y^2 \dot{x} r \).

2r. And by substituting in the room of \( x^2 \) or \( y^2 \), or of \( x \) or \( y \), their Values from the Equation of the Curve, there will be produced an Equation which the Fluent is the Length, or Area sought.

Thus in Case 1. If \( y = x^2 \), then \( \dot{\lambda} = x x^2 \), th. \( \dot{x} = \frac{x^{n+1}}{n+1} \frac{\dot{x} x^2}{n+1} \frac{1}{n+1} \) \( xy \). And the Asymptotic Space, in the Hyperbola, (\( n \) being there Negative) is \( \frac{1}{n+1} \). Therefore if \( n <, =, > 1 \), the Space will be Finite, Infinite, or More than Infinite.

More Examples are given where Occasion requires, th. are here, for Brevity's sake, omitted.

There are various other ways of finding the Lengths, or Areas of particular Curve Lines, or Planes, which may very much facilitate the Practice; as for Instance, in the Circle, the Diameter is to Circumference as 1 to

\[
\frac{16}{5} - \frac{4}{239} - \frac{4}{5} + \frac{16}{239} - \frac{4}{5} - \frac{4}{239} = \text{c.e.} = 3.14159, \text{c.e.} = \pi.
\]

This Series (among others for the same purpose, and drawn from the same Principle) I received from the Excellent Analyst, and my much Esteem'd Friend Mr. John Machin; and by means thereof, Van Ceulen's Number, or that in Art. 64. 38. may be Examined with all desirable Ease and Dispatch.

Whence in the Circle, any one of these three, \( a, c, d \), being given, the other two are found, \( a, d = c = \pi \)

\[
\frac{d}{2} \pi \frac{1}{x} x dx = d x = \frac{\pi d^2}{4} \pi d = \frac{1}{x} \pi x d^2 = \text{And.}
\]
And any Segment of a Circle = Sector = Triangle therein = $\frac{1}{2}ar = \frac{1}{3}sr$; But like Segments are as their respective Circles, i.e. $\pi r^2$ (the Circle where $r = 1$); $\frac{1}{2}d = \frac{1}{3}s$.

$\frac{s}{2\pi} = \text{Segment} = \frac{s}{2\pi} - s\times d^2 \times \frac{1}{4}$; but $s$ (of $\pi$ degrees) in parts of $a$ is

$\frac{1}{3} \pi n$, th. Segment = $\frac{1}{3} \pi n - s\times \frac{d^2}{1} \times \frac{1}{4}$

Or let $u = \text{Versed Sine or Height of the Segment};$ and $s = \text{Sine of} \ n \ \text{the double Co-Arc (in degr. and parts to the Sine)} = \frac{2u}{d} \ (\text{from the Table of Nat. Sines})$.

78. To find the Surfaces ($s$) of Solids generated by the Rotation of Planes about an Axis.

1. Where the Axis of Rotation is the Abscissa, $\frac{d}{y} = \frac{1}{x} + \frac{1}{x} \times \frac{1}{r}$. Examples.

In a Cone, $y = \frac{rx}{b}$, th. $\frac{d}{y} = \left(\frac{c}{r} \times \frac{rx}{b} \times \sqrt{\frac{r^2 + b^2}{b^2}} + \frac{x^2}{b^2}\right)$

$= \frac{cx}{b} \times \sqrt{\frac{r^2 + b^2}{b^2}} = \frac{cx}{b} \times \sqrt{\frac{r^2 + b^2}{b^2}} = \frac{cx}{b} \times \frac{1}{b}$, th. $s = \left(\frac{cx^2}{2b^2} \times l\right) \times \frac{1}{2} \pi d$ or $\frac{1}{2} \pi dl$ (when $x = b$).

Th. $\frac{1}{2} \pi d l = \frac{1}{2} \pi dr = l : r$. And if the Side of the Frustum of a right Cone $= b$, its Curv'd Surface will be $\frac{1}{2} \pi x x b$.

In a Sphere,

$y = \frac{2r x - xx}{\frac{1}{2}}, \ \text{th.} \ y^2 = \frac{r^2 x^2 - 2rxx^2 + x^3 xx^2}{2r x - xx}$

$\frac{r x - xx}{\frac{1}{2}} = \frac{r x - xx}{\frac{1}{2}} \ \text{th.} \ \frac{d}{y} = \left(\frac{c}{r} \times 2r x - xx \frac{1}{2}\right)$

$xx x x 2r - xx | \frac{1}{2} x x$. Th. $s = \pi d x$ the Surf. of the Segment) = (when $x = d$) $\pi d$ or $\pi d^2$ the Surf. of the Sphere.
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Th. \( \pi dx : \pi d^2 : : x : d \). And \( \pi dX - \pi dx : \pi d^2 : : x - x : d \). Hence the Curved Surfaces of Segments, or Frustums of Spheres, cut by parallel Planes, are equal to the corresponding Surfaces of the Sphere's Circumfr. Cylinder.

And \( \frac{2}{3} ed : ed : : \frac{1}{n} c : c, i : e. \) The Surface of the Sphere contained between any two great Circles, is to the whole Surface, as the \( \ell \) of Inclination, is to \( 4 \). 

Th. \( 1 : r :: 3Ls \) (of any Sph. \( \Delta \) abc) \(-2Ls \) (in parts of \( r \)): Surf. of the Sph. \( \Delta \). For \( \Delta be = seh, \) and \( \Delta be = sah \); And \( 4L : \ell \) \( \ell \) cba + bca + bac = \( \ell \) (Surf. Sph.): bahcb + cbiac + abeca, i : e, 4L : 3Ls = \( \ell \) (\( \ell \) c) : 3Ls - \( \ell \) (d') = \( \ell \) : 4\( \ell \), th. \( 2L \) (\( \ell \) a) = 2\( \ell \) c, Th. \( \Delta = \ell \) a d, th. & c.

2. Where the Axis of Rotation is parallel to the Abscissa, (let the longest Perp. from the Curve to the Axis of Rotation be \( p \)). \( \hat{s} = \frac{p - y}{\sqrt{x^2 + y^2 + 1}} \times \frac{x + 1}{3} \times \frac{x}{\ell} \). 

3. Where the Axis of Rotation is the Base of the Curve, \( \hat{s} = \frac{p - x}{\sqrt{x^2 + 1}} \times \frac{x + 1}{3} \times \frac{x}{\ell} \). 

4. Where the Axis of Rotation is a Tangent to the Vertex of the Curve, \( \hat{s} = \frac{y}{\sqrt{x^2 + 1}} \times x \times \frac{x}{\ell} \). 

79. To find the Contents (\( \chi \)) of Solids generated by the Rotation of Planes about an Axis.

1. Where the Axis of Rotation is the Abscissa, and the generating Plane adjacent to it, or the Tangent at the Vertex. (1) \( \chi = \frac{xy^2}{\ell} \times \frac{c}{d} \) d, or (2) \( \chi = \frac{x^2y}{\ell} \times \frac{c}{d} \) d.

Examples in Case 1. of this.

In a Cone, \( y = \frac{rx}{b} \), th. \( y^2 = \frac{r^2x^2}{b^2} \), th. \( \chi = \frac{cr^2x^2}{bd^2} \), th. \( \chi = \frac{cr^2x^3}{3db^3} \). Thereof. Cylinder \( (\frac{2}{3} \pi d^2 b) : \text{Cone} (\frac{2}{3} \pi d^2 b) :: 3 : 1 \).

Or because the Elements of a Pyramid, or Cone, are as the Sq. Numbers from \( o \), th. their Sum, i.e. the Pyramid, or Cone, is \( \frac{1}{3} \) of so many times the greatest, i.e., for
of the Circumscr. Prism, or Cylinder. Or sup. B the Base of a $\Delta$, $b$, $b$, &c. its Parallels, $B \equiv b = \tau$, or $\delta$, Th. $b = B - \delta$, and $b^2 = B^2 - 2B\delta + \delta^2$, th.

all the $b^2$s $= bB^2 - bbB + \tfrac{1}{2}b\delta^2 = bB^2 - B\delta + \tfrac{1}{2}\delta^2$

$= b \times Bb + \tfrac{1}{2}\delta^2 = \tfrac{1}{2}b \times 2Bb + \delta^2 (B - b \tau) = \tfrac{1}{2}b \times B^2 + Bb + \delta^2 = \tfrac{1}{2}b \times \tau^2 - Bb = Frustum$ of a sq. Pyramid.

Th. the Frustum of any Pyramid or Cone, $(A, a$ being the Areas of the Bases) is $b \times A + \sqrt{Aa} + a \times \tfrac{1}{2}\tau$. If the Bases are Circles, $D$, $d$, their Diameters, then the Frustum $= b \times \tau^2 - Ddx + \tfrac{1}{2}\pi$, and the whole Cone $= \tfrac{1}{2}\pi D^2 b$, as before.

If the Frustum of a Cone be cut by a Diagonal Plane passing thro' the op. Extremities of the Bases, then the

Greatest, or Least Frustum $= b \times \tfrac{1}{2}Dd + D^2$ or $d^2 \times \tfrac{1}{2}\pi \tau$.

This is evident, as also, the preceding Theorems, upon Sight of a Frustum of a Right Pyramid of Four (or any even Number of Sides) reduced into its Component Pyramids and Prisms.

In the Sphere, $y^2 = dx - x^2$, th. \( (dxx - cx^2 + d) \)

\[ = cx - cx^2 + \frac{1}{2}d, \]

th. \( \chi = \frac{1}{2}cx + \frac{1}{2}d = \frac{1}{2}\pi dx^1 \)

\[ - \frac{1}{2}\pi x^2 = \text{(when } x = d \text{)} \frac{1}{2}d^2 \text{ or } \frac{1}{2}\pi x^3 = \frac{1}{2}dx \text{ Surf.} \)

\[ (\pi d^2, \text{)} Th. Sph. (\frac{1}{2}\pi d^3) : Cylind. (\frac{1}{4}\pi d^3) :: 2 : 3. \]

Or since the Ring DE (\( \odot CE \equiv \odot AE \equiv \odot CA \equiv \odot AN \)) all the Rings DEs = all the \( \odot ANs \), th. Figure YEXZY = Cone CXZ = \( \frac{1}{3} \text{ Cylind.} \And \text{Hemisph. (Cyl.} \text{ Cone) = } \frac{1}{4} \text{ Cylinder. Th. Cone, Sph. and Cylind. are as } 1, 2, \text{ and } 3. \)

Or since \( s^2 = c^2 (dv) - v^2 \), th. all the s's = \( \frac{1}{2}dv \times c^2 \)

\[ = v \times \frac{1}{2}v \quad = \left( \frac{1}{2}v \times s^2 + v^2 \right) (c^2) - \frac{1}{2}v \times \frac{1}{2}v \times v^2) = \]

\[ = \frac{1}{2}v \times s^2 + \frac{1}{2}v^2 \text{ or } \frac{1}{2}v \times s + \frac{1}{2}v \times v, \text{ or } v \times 2s \left( \frac{1}{2}v^2 + \frac{1}{2}v \times \pi, \right. \]

or \( v \times 2s \left( \frac{1}{2}s + \frac{1}{2}v^2 = Segment \right) \text{ of a Sphere. Th.} \)

\[ \text{Spb.} = \frac{1}{2}\pi d^2 r, \text{ and Spb. } = \frac{1}{2}\pi d^2, \text{ as before.} \]

And
And, if a Sphere be cut by any two Parallel Planes, $S_1, S_2$, shall be the Radii of the Frustum's Bases, its Height ($V - v$) call $b$, then $V = b + v$; but bec. any Segment of a

$\text{Sph.} = \pi r^2 - \frac{1}{3}\pi v^3$, Th. Frust. $= 2rv - v^2 \times \pi b + \frac{r - v \times \pi b}{r - v} = \frac{S_1 - S_2}{S_1 + \frac{1}{3}v} = \left(\text{bec. } 2rv - v^2 = s^2, \text{ and } r - v = \frac{S_1 - S_2}{S_1 + \frac{1}{3}v}\right) \frac{S_1 + \frac{1}{3}v \times \frac{1}{3}v}{S_1 + \frac{1}{3}v} = \frac{S_1 - S_2}{S_1 + \frac{1}{3}v} \times \frac{1}{3} \pi$. Th. $v \times \frac{s^2}{1} \times \frac{1}{3}v \times \frac{1}{3} \pi = \text{Segment, and } \frac{1}{6} \pi d^2 r = \frac{1}{3} \text{ Sphere}$, as before.

In an Oblong Spheroid, let $a = \frac{1}{2} t = \frac{1}{2}$ the Transverse, $r = \frac{1}{3}$ the Conjugate Axe, then (bec. $a^2 : r^2 : 2ax - x^2 :$

\[ y^2 = \frac{2axr^2 - x^2r^2}{a^2} \] \[ \chi = \frac{2axr^2 - x^2r^2}{a^2d} \] \[ \text{Segment, th. } \frac{2r^2a}{3d} = \frac{1}{6} \pi d^2 \pi a = \frac{1}{6} \pi d^2 \pi a \]

Spheroid, and $\frac{1}{6} \pi d^2 \pi a = \text{Spheroid. Th. Cylind. (} \frac{1}{2} \pi d^2 t) : \text{Spheroid (} \frac{1}{6} \pi d^2 t) : : 3 : 2$. And Cone (\text{Cylinder}) $= \frac{1}{3} \text{ Spheroid}$.

Or (sup. Parameter $= p$, and $p - t = \pi$) since $y^2 (\pi \times 2ax - x^2) = \pi x a^2 - ax^2$ or $\pi x a^2 - \pi m^2$, th. all the $y^3 s = \frac{b^2}{3} x a^2 - \frac{1}{3} m^2 = \frac{1}{2} b^2 x a^2 - \frac{1}{3} m^2 = \frac{1}{2} b^2 x a^2 - \frac{1}{3} m^2 = \left(\text{bec. } y^2 = \pi a^2 - \pi m^2, \text{ and } r^2 = \pi a^2 \right) \frac{1}{2} b^2 x a^2 - \frac{1}{3} m^2$, Th. $\frac{1}{2} b^2 x a^2 - \frac{1}{3} m^2$ or (if $D, d$ be the Diameters of the Bases) $b \times 2D^2 + d^2 \times \frac{1}{3} \pi = \text{ Frust. of a Spheroid }$ cut by two parallel Planes, the one palling thro' the Centre. Th. the Spheroid $= D^2 r \times \frac{1}{3} \pi$, as before.

If $y = x^2$, then $\chi = \frac{1}{d} \frac{c \cdot x^3}{2m + 1} \frac{x^3}{4r}$, and $\chi = \frac{c x^{3 + 1}}{2m + 1} \frac{x^3}{4r}$.

Th. if $m = \frac{1}{2}$, $\chi = \frac{c x^{3 + 1}}{2d} + \frac{c x^{3 + 1}}{4r}$, $\frac{1}{2} \pi erh = \frac{1}{2} b \times \frac{1}{3} \pi = \text{Obtuse Parabolic Conoid, Th. Conoid (} \frac{1}{2} \pi d^2 b) : \text{Cylind. (} \frac{1}{6} \pi d^2 t) : : 1 : 2$. And Cone, Conoid, and Cylinder are as 2, 3, and 6.
2. Where the Axis of Rotation is the Base (b) of the Curve, and the Plane adjacent to it, or to the Tangent at
the Vertex (1) \( \chi = \frac{x}{r} y - xy \times \frac{c}{r} \times \frac{1}{r} b \), or (2) \( \chi = \frac{x}{r} - \frac{c}{r} \times \frac{1}{r} b \).

Example in Case 1. If \( y = x^n \), \( \chi = px^n x - \frac{cx^{n+1}}{r} \times \frac{c}{r} \), th. \( \chi = \frac{cx^{n+1}}{r \times n+1} - \frac{cx^{n+2}}{r \times n+2} \). Th. if \( n = \frac{1}{2} \), as in the

common Acute Parabolic Conoid, then \( \chi = \frac{cx^2 y}{r} - \frac{cx^{2y}}{r} = \frac{1}{2} \pi d^2 y - \frac{1}{2} \pi d^2 b \). Th. is to the Cylinder as
8 to 15.

Or in this Solid, call the Axe of the Curve B, its Parallels b, b, &c. and B - b = \( \delta \). Th. \( b = B - \delta \), and \( b^2 \)
\( = B^2 - 2B \delta + \delta^2 \), th. all the \( b^2 s = bB^2 - \frac{1}{2} b \times 2B \delta + \frac{1}{2} b \delta^2 = \)

\( b \times B^2 - \frac{1}{2} b \times B \delta + \frac{1}{2} B \delta^2 = \frac{1}{2} b \times B^2 + \frac{1}{2} B \delta + \frac{1}{2} B \delta^2 \) or \( \frac{1}{2} b \times 2B + \frac{1}{2} B \delta + \frac{1}{2} \delta B \). Frukt.
of an Acute Parabolic Conoid, \( = (if D, d be the Diameters of the Bases) \frac{2D^2 + d^2 - \frac{3}{2} D^2 \times b \times \frac{1}{2} \pi} \).

3. Where the Axis of Rotation is a Tangent at the Vertex of the Curve, and the Plane adjacent to it, or to the Ab- 
scissa. (1.) \( \chi = y \times x \times \frac{c}{r} \delta \), th. \( \chi = xy \times \frac{c}{r} \times \frac{1}{r} b \). So
that (in Case 2.) if \( y = x^n \), \( \chi = \frac{cx^{n+2}}{r \times n+2} \), and \( \chi = \frac{cx^{n+2}}{r \times n+2} \), th. if \( n = \frac{1}{2} \) (as in the Common Parabola)

\( \chi = \frac{2cx^2 x}{2} = \frac{1}{2} \pi d^2 b \); Th. is to the Cylinder as 4
to 5. And the Solid generated by the Infinite Asymp-
totic Space (in the Apollonian Hyperbola, where \( y = x^{-1} \))
about that Asymptote, is \( \frac{1}{2} \pi d y \); Th. is to the Cylinder
(the Rad. of whole Base is \( x \), and Alt. = \( y \)) as 2 to 1. So
that, in this Case, an Infinite Space generates a Solid of a 
Finite Dimension.

4. Where
4. Where the Axis of Rotation is parallel to the Abscissa, and the Plane adjacent to it, or to the Abscissa; (1) \( \chi = x^2 - y^2 \times \frac{c^2}{d} \), or (2) \( \chi = x^2 - y^2 \times \frac{c^2}{d} \).

Note. Where the Ordinates are Oblique to the Diameter, or Axis of Rotation, or where they respect a determined Point; they are to be reduced to other equivalent ones, parallel between themselves, and right to the Axis of Rotation.

Hence in Cask Gauging, let \( B, H \) be the Bung, and Head Diameters, \( L \), the Length, \( n \) the Number of Cubic Inches in the Gallon; and the Diam. : Periph. :: \( 1 : \pi \).

Then the Contents of a Cask taken as the middle Frustum of an Acute Parabolic Conoid, or of an Oblong Spheroid, is

\[
L \times 2B^2 + \frac{H^2}{3} - \frac{1}{3}B - H \div \frac{12\pi}{\tau} \quad \text{or} \quad L \times 2B^2 + \frac{H^2}{3} = \frac{12\pi}{\tau}
\]

Th. in a Spheroidal Cask, when not Full, the Axis being Perpendicular to the Horizon, \( w \) = wet part of \( L \), \( d = \frac{1}{2} L \); if \( w \) be >, or < \( \frac{1}{2} L \), the Cask is more, or less than \( \frac{1}{2} \) Full by \( 3B^2d - d^3 \times B^2 - 3L^2 \div 4L^2 = \frac{12\pi}{\tau} \), and therefore the Liquor contain'd therein must be

\[
\frac{1}{2} L \times 2B^2 + \frac{H^2}{3} \pm 3B^2d + d^3 \times B^2 - 3L^2 \div 4L^2 = \frac{12\pi}{\tau}
\]

But in a Cask not Full, whose Axe lies parallel to the Horizon, and the Liquor cutting the Head; Let \( w = \) wet part of \( B \), \( S = \) Segment of a Circle (whole Area = 1) to the verfed Sine \( w \div B \), and \( C = \) Content of the Cask, then will SC be the Quantity of Liquor remaining.

8c. Of Projection.

The Impression or Representation of a Surface, on a Plane is called the Projection of that Surface: If the Impression be made by parallel Lines, 'tis called an Orthographic Projection; But if by Lines intersecting in the same Point, 'tis called a Stereographic Projection.
1. In the Orthographic Projection, 'tis evident that, 1. A right Line Perpendicular, Parallel, or Oblique to the Orthographic Plane is projected into a Point, an Equal, or a Shorter Line: And parallel Lines, tho' oblique to the Plane, are parallel when projected. 2. A Surface perpendicular to the Plane is projected into a right Line, equal to that intercepted between the projecting Lines: But a Surface parallel to the Plane, into an equal and like Surface. 3. Any Conic Section Oblique to the Plane is projected into another of the same Kind; Hence a Circle into an Ellipse. 4. The Representation of any Point of the Surface of the Sphere, Orthographically projected upon a Plane bisecting it, is distant from the Centre, by the Sine of the Arc from the Point to be projected to the Vertex of that Sphere.

Hence $ca$ is Equivalent to $rs$, $cb$ to $ad$. And all Circles Perpendicular, or Parallel to the Plane, are projected into Right Lines, or Equal Circles; but all other into Ellipses. Thereof all right Lines passing thro' the Centre of this Projection, are measured from the Sines; and all others are reduced to such; As for instance,

Given the Latitude of the Place, Sun's Declination and Altitude; Req'd the Sun's Azimuth, and Hour of the Day; the Sphere being Orthographically projected upon the Plane of the Meridian.

Let $va = \text{Co-Latitude}, ra \text{ or } ca = \text{Sun's Altitude}, ad \text{ or } cb = \text{Declination}; \text{Draw } aa, \text{do the Parallels of Alt. and Decl. draw } oo, \text{ such } || \text{ ac, cb; then } cz, cu = \text{Sines of the like Azimuth from the West, or Hour from 6, } \odot, ca, \odot.

To describe the Representation of Oblique Circles on this Projection; As suppose, to describe a Meridian, passing thro' the nth. Degree of the Equator, on the Alemma.

'Tis but making $ca : cn : : \chi v : \chi r : : 6d : 60$, and the Points $v, \odot$, are in the Ellipse found. Or the whole Ellipse (according to the Method in Art. 68.) is readily drawn by means of the Foci, $(0, t)$ found by cutting the Transverse Axe from $n$ with $pe$ its half.
In the Stereographic Projection, that point in which all lines from the extremes of the thing projected do concur is called the Projecting Point; those lines the Projecting Lines; the plane cutting them, the Stereographic Plane, or Plane of Projection. And the Stereographic Plane is supposed (if not otherwise mention'd) perpendicular to another which is also supposed parallel to the horizon, having thereon, the sect or Isomography of the thing to be projected, and called the Original or (by some) the Geometrical Plane. A plane passing thro' the projecting point, \( \parallel \) to the horizon perp. to it and to the Pl. of Proj. is call'd the Horizontal, or Vertical Plane. The corner. Sect. of the Horizon Plane, and Plane of Proj. is called the Horizontal Line, and the corner. Sect. of the Vertical and Horizon Plane is called the Axis of Projection. The Intersect. of the Axis of Projection and the Horizontal Line is called the Centre of Projection. The corner. Sect. of the Stereographic and Original Plane is called the Base-Line. The Intersect. of the Horizon Line, and another drawn from the projecting point parallel to a given right Line \( (L) \) is called the Projecting Centre of the Line \( L \).

In this Projection, 'tis manifest, that, 1. The Projection of a Point, or Right Line is a Point, or Right Line. 2. The Projection of a right Line, on or above the Original Plane, parallel to the Base-Line is parallel to it in the Pl. of Projection: And the parts of its Projection, are : : : to those of the line they fell. 3. The Projection of a right Line \( L \) or inclin'd to the Original Pl. and \( \parallel \) to the Pl. of Proj. is \( L \) or so much inclin'd to the Base Line: And the Projections of the equal parts of a Line perp. to the Pl. of Proj. are unequal; and those of the most remote are the smallest. 4. The Projections of parallel right Lines on or parallel to the Orig. Pl. and inclin'd to the Plane of Proj. being produced, shall all pass thro' the same point in the Horiz. Line. 5. Thereof, the Projection of right Lines perpend. to the Pl. of Proj. do pass thro' the Centre of Projection. 6. The Projections of equal Lines perp. or equally inclin'd the same way, on the same right Line \( (L) \) inclin'd to the Pl. of Projection, are limited by two Lines, which being prolong'd, shall pass thro' the Projecting Centre of the Line \( L \). 7. The Projections of Points above, or below the Horiz. Plane, are above, or below the Horizontal Line; and the more remote
remote they are, the lower, or higher they be. 3. Given, the Height of the Projecting Point above the Original Plane, with its Distance from the Plane of Projection, as also the Diff. of the Points to be projected from the same; Reqd. the Projection of that Point. 1. If the Point to be projected be on the Original Plane; Let E, be the Proj. Centre, & B, C, the Plane, and Centre of Projection; Pa Point in the Orig. Plane A, whose Projection p is sought. Set EC on the Horizont. Line, and PN on the Base-Line, the contrary way, as from C to d, and from N to n, draw nd, CN, their Intersect. p is the Projection of p reqd.

For (bec. Ed || Pn) d is the Projecting Centre of the Line Pn, but the Project. of P is in nd (by 4) and in NC (by 5) th. must be in their Intersect. p. 2. If the Point to be projected be (a) above the Original Plane (A) Find P its Seat, and N the Point of Incidence of P, ereft Na = Pa, then will CN, Ca be the produced Projections of PN, sa (by 5), and p the Project. of P (by the left.) th. (drawing pπ || Na) π will be the Project. of a sought.

These are the Principles on which the Practical Part of Common Perspective depends; and by which 'tis easie to give the true Represen[tation of any Object, whose Situ-
Situation, with Respect to the Eye and to the Plane on which 'tis to be represented are, given.

As in the following Instance, where 123 is the Base of a Triangular Prism, whose Altitude is bt, to be put into Perspective; c the Centre of Projection or Points of Sights; d the Projecting Centre or Points of Distance: The Præ-

stice is evident from the foregoing Rules.

Note, That 'tis indifferent what Point of the Horizontal Line be taken, for determining the Elevations; it may as well be at n as at c; for zx || a3, bec. nx : xt : : (ca : at : : c3 : 3h : : ) nz : zh, th. zx (|| ht) || a3.

9. The Stereographic Projection of any Point (A) of the Surface of a Sphere, from the Pole of a Plane bisecting it, upon that Plane, is distant from the Centre by the 1/4 Tangent of the Arc from the Point to be projected to the Pole opposite the projecting Point; Thus, M is the Projection of A; th. BM equiv. BA, MC equiv. Ao, and cA equiv. Md.

1. Here all Circles passing thro' the projecting Point are projected into Right Lines; But all others into Circles: Since the Plane of the Projection cuts the Surface of the Cone form'd by the projecting Lines, either parallel, or (bec. ∠BQP = 1/4 arc BP = 1/4 arc ba (by 54) = 1/4 arc Pa = LAAP) contrary to its Base. And the Angles in this Projection are equal to their correspondent ones on the Sphere.
2. If AA be the Diameter of a great Circle to be projected, whose Obliquity is AB (or ω); let 0x = AB = πk, draw PPx, Pk, p shall be the Upper Pole (bec. πA = 90°) and c the Centre of the Projected Oblique Circle PMO; since cM (= t, ω + t), \(\frac{1}{2}90° - \frac{1}{2}\omega = 1, \omega\) (by 65.16) = cP. And if Pk = AB or OX, a right Line thro' P and k will project the Under Pole in the produced Plane.

Also the Centre of a Lass Circle (Ly) perpendicular to the Plane, is found by drawing CL, and Lu \(\perp\) to it, then will u be the Centre reqd. Since uy (= l, \(\omega - t\),

\(\frac{1}{2}90° - \frac{1}{2}\omega = t, \omega\) (by 65.16) = Lu.

The Radius and Centre of any Circle whose Distance from its Pole is \(d\), may easily be found thus: From (P) the projected Pole, on a great Circle (whole Centre is c) passing thro' it, \(d\) being let off, finds a Point (P), whose Tangent, terminated at x in a Line thro' c and P, or (which is the same) a Tangent to P, terminated at x in a Line thro' c and P, shall be the Radius, and \(\pi x\) the Distance from the Centre required.

3. The Centre x of any great Circle (NM) passing thro' M, shall be in \(\infty\), the produced Diameter of a Circle (PMO) bisecting the Primitive in P and o; And (bec. Sph. \(L PMN = L CMx\) by 56) \(cC\) = Tangent of the Sph. \(L PMN\) to the Rad. Mc.

4. Hence to draw the Circums. of a great Circle thro' any two Points, (m, n), one, or both within the Primitive, Thro' (m).
Palmariorum Matheseos.

(m) one of the Points in the Circle, draw a Diameter $AA$, and $AB$ to it, draw $= 2AR$, a Ruler on $x$ and $f$ cuts $AA$ in $e$, thro' $e$ draw $ef // AB$, a right Line ($Bf$) bisecting the Distance between $m$ and $n$ at $l$, will cut $ef$ in the Centre sought. For the Centres of all great Circles passing thro' $m$ are in the Line $ef$, and the Centre of a Circle passing thro' $m$ and $n$ is in the Line $Bb$, th. the Centre reqd. must be in both, i. e. in their Intersection $e$.

5. The Circumf. of a great Circle in this Projection, is Divided, or Measured, by a Ruler laid on either Pole, and the Divisions of the op. $\frac{1}{2}$ Periphery of the Primitive; For, since $Bo$ equiv. $Mo$ equiv. $Co$ equiv. $ce$ equiv. $90^\circ$. th. $BD$ (equiv. $Cy$ equiv. $cf$) equiv. Md. And the Circumf. of a small Circle (whole Dist. from its Pole is $d$) may be divided by Lines from that Pole to the Divisions in the Circumf. of a small Circle $||$ to the Pl. of Proj. and desc. at the Dist. $d$ from the under Pole of that Plane.

Or any Projected Circumf. is divided by Lines from the Divisions of its produced $\frac{1}{4}$ Periphery to either Pole.

6. A Spheric $\Delta$ or the Inclination of the Planes of two great Circles is equal to the Distance of their Poles, or (which is the same) is measured by an Arc of a great Circle whose Pole is the Angular Point; and thereof, is so in the Projection. And the Circumsfs. of great Circles intersecting at $l$, do mutually pass thro' each other's Poles.

7. The $\Delta abd$, and $\Delta add$ are the Stereographic and Orthographic Projections of the same Sph. $\Delta$; and their $\Delta s$ $b$ and $d$ are in the same Radius $cu$. Let $ss$ perp. $cu$.

Given $b$, Req'd. $d$; draw $b$, and $vd || ss$.

Given $d$, Req'd. $b$; draw $dv || ss$, and $vbs$.
8. Let $\nu = \nu$, draw $SAQ$, with $q$, from $q$, describ. a Circumf. $S$ cutting $cT, cR, db, ab$, in $p, \tau, z, w$, then will $p, \tau, c$, be the Pole of $db, ab, ad$; and $b$ the Pole of $wS$; draw $bpe, bTc$, then $Ec = \tau p = Sph. Lb$.

9. Draw $A\alpha m, GpM$, make $mn = ma$, and $MN = Md$, a Line thro' $A$ and $n$, $G$ and $N$ gives $k, K$, the Centre of the Arc $aAbA$, $dbG$.

10. Draw $A\alpha B = HR, eT$, then will $H\alpha, \beta$ equiv. $bX, b\chi$.

11. And in the Sph. $\Delta c\nu p, Lc = \text{ad}; Lp = \chi z = db$ equiv. $db\beta$; $Lc = \text{Supl. xw} = \text{Supl. ab} = \text{Ab equiv. Aa, i.e.}$

\[
\begin{align*}
&\text{in } \Delta c\nu p, \text{ is } = \{\text{ad, db, Supl. ab}\} \\
&\text{in } \Delta abd.
\end{align*}
\]

12. Hence (bec. $ad + db + Aa = 2 Ls + a\beta$) the $3 Ls$ of a Sph. $\Delta$ is greater than $2 Ls$.

10. The Projection of any Point ($P$) of a Hemisphere, from the Centre ($C$) of the Sphere, upon a Plane touching it (in $T$) is distant from the Point of Contact ($T$) by the Tangent of the Arc $PT$; and this is called the Gnomonic Projection; where 'tis plain, that, 1. All great Circles on the Sphere, perpendicular to the Plane, are projected into Right Lines; all others, (as they are parallel to, touch, cut, or neither, a great Circle parallel to the Plane) into Circles, Parabola's, Hyperbola's, or Ellipses.

Hence, if the Sun moves in a Circle parallel to the Equator, the Shade of the Centre will describe a Conic Surface; And therefore, upon a Plane parallel to a great Circle, touching, or cutting the Conic Surface, or neither, will describe the Curve of a Parabola, Hyperbola, or Ellipse. Therefor. on any Plane where the Sun sets, does not set, or only touches, the Projection of any Parallel of Declination is an Hyperbola, Ellipse, or Parabola.

Thus, if $ce, ca$ be the Stile, and Substile, Sec or desc. the Co-declinas. of the Sun; then,

If $ed$ produced be parallel to the Plane, and $ed$ produced cut it in $q$, let $en = eq$, from $n$ with $nq$ cut $qc$, the Projected Meridian produced, in Sup. $\tau$, then $\tau q = \text{Parameter}$, and $\frac{1}{2} \tau q$, set from $q$ the Vertex, gives the Focus $f$ of the Parabola. If $ed$ produced cut $ca$ in $q$, and $ed$ produced cut it in $Q$, above, or below the Vertex $e$, let $en = eq$, bisect $Qq$ in $y$, make $yf = y\phi = \frac{1}{2} nQ$, then shall $f, \phi$ be the Foci of the Ellipses, or opposite Hyperbola's.
Bola's described by the Shade of the Point $e$, their Transverse Axe being $Q$ and Distance of the Foci $nQ$ (by 69.6).

Hence the Parallels of the Sun's Declination, &c. in the Gnomic Projection, are easily described (by 68.) upon an Horizontal Plane, in any Latitude.

And all Planes are parallel to some great Circles, which also are Horizons in some place or other; therefore all Planes on which Dials are made, are Horizontal Dials on some part of the Earth.

The Method of projecting the Hour Circles Gnomonically, on an Horizontal Plane, &c. or the Practice of Dyalling is hence also evident; Thus, let $c12$ be the Projection of the Meridian, and $(c61)$ to it, that of the 6th Hour Circle; make $Lace = Stile's Heighe$, and $af =$ Sinne thereof, to any Rad. $c4$; let $ax = cm = af$, on $ax, mx$, from $a, m$, let the Tangents of the Hour $L$ at the Pole, from 12 to 3, and 6 to 3, to the Rad. $ax, mx$; Lines drawn
drawn from th'o' those Points, shall be the Hour-Lines reqd.

For $\frac{1}{2} ca : r :: ax : t, \text{ acx. and cm : r :: mx : t, mcx}
\frac{1}{2} ca : r :: ax : t, \text{ acx. and gm : r :: mx : t, mgx}
th. ca : fa :: t, \text{ afx : t, acx. and cm : gm :: t, mgx : t, mcx}
i.e. \text{ Rad. : S, Stile's Height :: t, Hour L as the Pole : t, Hour L on the Plane.}

81. Of Trigonometry.

The Mensuration of the Sides and Angles of Triangles, whether Plane or Spherical, is called Trigonometry.

A Triangle has Six Parts, viz. 3 Sides, and 3 Ls, whereof any three (except the 3 Ls of a Plane $\triangle$) being given, the other three may be found by Trigonometrical Calculation.

1. Of Plane Trigonometry.

PROBLEM 1. In any right-angled $\triangle$, if among the Data, there be an opposite Side and Angle.

RULE 1. As any one Side is to the Sine of its op. L. So is any other Side to the Sine of its op. L.

For half the Sides are the Sines of their op. Ls (by 51) but halves are as their wholes, th. the Sides are as the Sines of their op. Ls.

PROBLEM 2. Given two Sides and an L included; Req'd. the Ls. The Legs being $a, c$; and $y$ the incl. L; then $180^\circ - y = \gamma = \text{Sum of the unknown Ls whose Difference call } d$.

RULE 2. $a + c : a - c :: t, \frac{1}{2} d : t, \frac{1}{2} d$; And $\frac{1}{2} a + \frac{1}{2} d =$, $< L$ reqd.

For $a + c : a - c :: t, \frac{1}{2} s, \frac{s + s}{2} : t, \frac{1}{2} d$ (by 66.2)

Note, If $Ly = L$, then $a : r :: c : t, \text{op. L,}$ or $c : r :: a : t, \text{op. L.}$

For the Tabular Parts of any two Lines are proportional to their Parts according to any other Measure.

PROBLEM 3. Given the three Sides; Req'd. the Ls. Let $a, c$ be the Legs of the L reqd. and $a^2 - c^2 = d$, or $d$, call the Base or Side op. $b$, and the Diff. of the Segments of $b$, made by a perp. from the $L$ sought, call $\lambda$.
RULE 3. \( \frac{b}{a} : \frac{c}{x} : : d : a \) (For \( p^2 + n^2 + \frac{a^2 + c^2}{b^2} \))

\[ \frac{1}{2} + 2 \frac{a}{b} = \frac{p^2 + n^2 + a^2}{b^2} \quad \text{Th.} \]

2. a + d :: d : a And \( \frac{a}{x} + \frac{d}{a} \)

> Segment; whence the \( L \)s may be found by Prob. 1.

Or bec. \( r : s \left( \frac{c}{x} : \frac{c}{x} \right) \quad 2 \frac{a}{c} : \left( 2 \frac{a}{c} \times x \right) = \frac{a^2 + c^2 + b^2}{b^2} \) (by 64. 17) th. \( 2 \frac{a}{c} : \left( \frac{a^2 + c^2 + b^2}{b^2} \right) r^2 : b^2 \), or \( b^2 = \)

\[ \frac{1}{2} + \frac{a}{b} + \frac{c}{x} \left( \frac{c}{x} + \frac{b}{c} \right) \quad \text{then will these Rules follow, viz.} \]

1. \( \frac{1}{2} + \frac{a}{b} + \frac{c}{x} \left( \frac{c}{x} + \frac{b}{c} \right) \quad \frac{ac}{b^2} : \left( \frac{ac}{b^2} + \frac{a}{b} \right) \quad b \times \frac{b}{c} : r^2 : s^2 : \frac{a}{c} \]

2. \( \frac{1}{2} + \frac{a}{b} + \frac{c}{x} \left( \frac{c}{x} + \frac{b}{c} \right) \quad \frac{bc}{b^2} : \left( \frac{bc}{b^2} + \frac{b}{c} \right) \quad b \times \frac{b}{c} : r^2 : s^2 : \frac{a}{c} \]

3. \( \frac{1}{2} + \frac{a}{b} + \frac{c}{x} \left( \frac{c}{x} + \frac{b}{c} \right) \quad \frac{bc}{b^2} : \left( \frac{bc}{b^2} + \frac{b}{c} \right) \quad b \times \frac{b}{c} : r^2 : s^2 : \frac{a}{c} \]

4. \( \frac{1}{2} + \frac{a}{b} + \frac{c}{x} \left( \frac{c}{x} + \frac{b}{c} \right) \quad \frac{bc}{b^2} : \left( \frac{bc}{b^2} + \frac{b}{c} \right) \quad b \times \frac{b}{c} : r^2 : s^2 : \frac{a}{c} \]

And if \( L = \text{Logarithm Sine, and} \ l = \text{Arith. Compl. of any Log. Sine, then, for Practice,} \)

\[ \frac{a}{b} + \frac{c}{x} + \frac{L}{s} + \frac{r}{b} + \frac{d}{a} = \frac{L}{a} \times \frac{b}{c} \]

To the various Uses of Plane Trigonometry, may be added, that,

If an Equation be \( x^3 + ax = b^2 \); Make \( b \), and \( a \) the Legs (AB, BC) of a \( \triangle \), Then Hypotenuse \( \pm \frac{r}{a} \), (i.e. EB, or BD) = x. Or bec.

\[ b : r : \frac{a}{x} : x \times \frac{c}{x} \quad \text{Th.} \quad r : b : s : s \times \frac{a}{x} : \]

If the Equation be \( x^3 + ax = b^2 \); Make \( b \), and \( a \) the Perp. and Hypot. (AB, AC) of a \( \triangle \), then Hypo-

tenuse \( \pm \text{Base} \) (i.e. BD, BE) = Greatest, or Least Root. Then bec. \( \frac{a}{x} : r : b : s \times \frac{a}{x} : \text{th.} \quad s : l \quad \frac{a}{x} : r \times \frac{a}{x} : \text{or} \quad s : l : a \) the \(< \text{or} > \text{Root.} \)

2. Of Spheric Trigonometry.

In \( \triangle \) Sph. \( \triangle \) CAB, Cab, Orthographically projected, sup. H, P, & B, the Hypotenuse, Perpendicular, and Base.
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*Synopsis*

**Part 2.**

1. \( \sum s, H:s, P:s, b:b, p:p \); \( \ell \) at the Base.

\[ \sum s, B:t, P:s, b:t, p:t, \] \( R:t \).

For \( cA : A \gamma = ca : a \); And \( cB : TB = cb : tb \).

2. In any Sph. \( \Delta (a:b) \), The Sines of the Sides are as the Sines of their op. Ls. For (by prec.) \( R : s, \beta \) : : \( b : s, \beta, \) \( Bb, \) and \( R : a : s, \alpha \) : : \( b : s, \alpha, \beta \).

3. Therefor. in Right Ld Sph. \( \Delta s \), of the three Parts which (besides the \( \bot \)) enter the Question, let that be called the Middle to which both Extremes are either Conjunct or Disjunct. Then, Rad. + Log. of the Middle = Logs. of the Extremes.

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<td>Sine.</td>
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Hence all the Caffes are easily solved.

For the Middle part may be one of the Legs \( (b, p) \) the Compl. of the Hypotenuse \( (b) \) or the Compl. of either of the Oblique Ls \( (e, e) \). In each, there are three Caffes: As sup. the Middle, be \( b, \) \( Le, \) \( b. \)

1. \( r : t, \ell \) : : \( s, b : t, p \) (by 1).

2. \( r : t, \ell \) : : \( s, a : t, e \) (by \( r \)).

3. \( r : t, \ell \) : : \( s, p : t, e \) (by \( r \)).

Th. \( R \times S \) Middle = Tangent of the Conj. Extremes.

1. \( r : (s, b) \) : : \( (s, \ell) \) = \( s, b \) (by 1).

2. \( r : (s, \ell) \) : : \( (s, \gamma) \) = \( s, \gamma \) (by 1).

3. \( r : (s, \gamma) \) : : \( (s, b) \) = \( s, b \) (by 1).

Th. \( R \times S \) Middle = Co-Sines of the Disj. Extremes.

Note, In L Sph. \( \Delta \), if the Legs (and th. the Ls) be like, or unlike; the Hypotenuse is \( \angle \), or \( > 90^\circ \); and the contrary.

If the Hypotenuse is \( < \) or \( > 90^\circ \), either Side will be like, or unlike its adjacent, \( L \), and the contrary.

In Oblique Ld Spherical Triangles; Suppose the Obl. \( \Delta \) to be divided into \( \angle \) \( \Delta s \), by a \( \parallel \) let-fall from one End of a given Side, (whose other End is adjacent to a given
given $L$ within, or without the $\Delta$, as the $L$s at the Base are of the same, or of different Kind. Then, if there are two parts given in one $\Delta$, corresponding to the two given and sought in the other, the perpendicular being a $3d.$ in each, will denote a Middle or Extreme Part; And (rejecting the Perp.) the Logarithms of the op. parts are equal. But if 4 such Parts don't correspond (as it happens when the perp. falls from or on a part given or required;) find (by 3) a part (in the $\Delta$ wherein a Side and an $L$ are given) of the given or sought one, from or on which the perp. falls; Then you'll have the 4 parts required for the Second Operation.

When two Sides, or $L$s, and the $L$, or Side included, are given; the other $L$s, or Sides may be found, without setting fall a perp. by these Proportions.

1. $s_1s_2:\theta_1\theta_2 = s_1s_2:ct\theta_1\theta_2 = t_{\theta_1\theta_2}$.

2. $cs_1s_2\theta_1\theta_2 = cs_1s_2\theta_1\theta_2:ct\theta_1\theta_2 = t_\theta_1\theta_2$.

For, let $v\beta\theta$ be the Oblique $sph. \Delta$ Stereographically projected, if the Right Line $b\beta$, and the Tangent $sb$, $b\alpha$, be drawn, it appears that the $sph. L$ $v\beta\theta = \text{Plane } L v\beta\theta = \text{Plane } (\theta_1\theta_2 : d_1d_2 : ) = \frac{\sin \theta_1\theta_2}{\sin \theta_1\theta_2} = \frac{\sin \theta_1\theta_2}{\sin \theta_1\theta_2}$ (by 81, 1, Pr. 2.) $: \frac{s_1s_2\theta_1\theta_2}{s_1s_2\theta_1\theta_2} = \frac{s_1s_2\theta_1\theta_2}{s_1s_2\theta_1\theta_2}$ (by 86.) And because $d_1d_2\theta_1\theta_2 = d_1d_2\theta_1\theta_2 = d_1d_2\theta_1\theta_2 = \frac{\sin \theta_1\theta_2}{\sin \theta_1\theta_2}$ (by 81, 2, 1.) then (by 86) $t_{\theta_1\theta_2}d_1d_2\theta_1\theta_2 = \frac{\sin \theta_1\theta_2}{\sin \theta_1\theta_2}$ (by 85) $t_{\theta_1\theta_2}d_1d_2\theta_1\theta_2 = \frac{\sin \theta_1\theta_2}{\sin \theta_1\theta_2}$, Th. $cs_1s_2\theta_1\theta_2 = cs_1s_2\theta_1\theta_2 = ct\theta_1\theta_2 = t_\theta_1\theta_2$. This Demonstration, with several other New and Valuable things of this Nature, I had from the Excellent Geometer Mr. Halley, whose Freedom in Communicating, and Readiness in Assisting, I shall always own with the highest Gratitude.

4. The two Cases excepted, viz.

1. Three sides being given; the Angles required. Let $x_1, x_2$ be the Legs of the $L$ sought, and $b$ the Base, or Side opposite.

Then $x_1^2 + x_2^2 + L_1^2 + L_2^2 = 2 = \log s_1s_2\theta_1\theta_2$.

$s_1s_2\theta_1\theta_2 = s_1s_2\theta_1\theta_2 = \phi \times \chi \times \psi$; $a\phi = a\phi + a\phi$; or equiv. $\phi \chi \psi = \log \omega m$; $me (m^2\omega = s_1s_2\theta_1\theta_2 \log \omega m) = s_1s_2\theta_1\theta_2$.

Draw $e = \omega \theta_1\theta_2 = (s_1s_2\theta_1\theta_2) = \phi \times \chi \times \psi$. Then $s_0s_2 = s_0s_2 \times \phi \chi \psi = \frac{1}{2} \omega \theta_1\theta_2$.

That is $s_0s_2 \times \phi \chi \psi = \frac{1}{2} \omega \theta_1\theta_2$.

$S_0 \times S_0$.
Synopsis

Part 2.

s, xx s, r : s, \frac{b}{2} \omega x + \frac{r}{2} x s, \frac{b}{2} + x + \frac{r}{2} : r^2 : \cos \frac{s}{2} \frac{1}{L} v.

And after the like manner 'tis prov'd that

s, xx s, r : \frac{b}{2} \omega x + \frac{r}{2} x s, \frac{b}{2} + x + \frac{r}{2} : r^2 : \sin \frac{s}{2} \frac{1}{L} v.

2. Three Angles being given; the Sides required. Instead of the greatest \( L \), take its Supplement, and call the \( L \)s Sides, and the Sides \( L \)s; then do as in the preceding Case.

82. The Principles of Mechanics.

1. Velocity \( v \) or that Affection of Motion, whereby a Body runs a given Space \( s \) in a given Time \( t \) is the Ratio of the Space to the Time, i.e. \( v = s \div t \); Th. \( vs = s \), and \( Vts = uvS \). Th. if \( V \), \( v \), \( t \), \( s \), \( sT^2 \), or \( sV^2 \).

2. Moment \( m \) or that which conduces to the effecting of Motion, is compounded of the Velocity \( v \) and Quantity of Master or Weight \( w \); i.e. \( m = uv \); Th. \( mV = Mun \). And bec. \( v = \frac{s}{2} - \frac{T}{2} = \frac{M}{2} \times W \), th. \( TM : sm :: WS : ws \).

That Motion is said to be Equable, which runs over all the Parts of Space with the same Velocity; but Accelerated, or Deferred when its Velocity is continually Augmented, or Diminished. The Innate or Natural Force of a Body is that by which it endeavours to preserve in its State of Rest, or uniform direct Motion. An Impress'd Force is an Action exercis'd on a Body to change its State of Rest, or Motion. Centripetal Force is that by which a Body is Impell'd, or Attracted towards some Point as a Centre. Centrifugal Force is that by which a Body endeavours to recede from its Centre.

3. All Bodies will continue in their State of Rest, or uniform direct Motion, unless they are compelled to alter that State by some Force impress'd upon them.

4. The Change of Motion is ever proportional to, and its Direction is in the same right Line with the Impress'd Force.

5. The Actions of two Bodies upon one another are always equal, and have contrary Directions.

6. Hence, if a Body in (in Fig. 6.) be impell'd by two different Forces \( F, t \) to move (in \( g, g, \)) with an uniform Velocity, it will describe the Diagonal \( gh \) of a Parallelogram.
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In the same time as it would describe the sides \((g\theta, g\lambda)\) by \(F, f\), separately, for, because the force \(F, f\) (in \(g\theta, g\lambda\)) hinders not the velocity of the force \(f\) from carrying the body \((g)\) to a line \((\lambda h, \theta h)\) parallel to the direction of \(F, f\), in the same time, whether the force \(F, f\) be impressed or not; th. \((g)\) will be found in each of the parallels, i.e. in their intersection.

7. Whence we have the Method of Compounding and Resolving any given Directions. For any Motion may be considered as compounded of others, and th. may be resolved into them.

8. If a body be held immoveable by two equal forces acting with contrary directions, since either of these forces may be resolved into two others, therefrom is the same as if the body was held by three different forces; and these three forces are one to the other, as lines drawn parallel to the respective directions, and terminated at their mutual concourse.

9. Also the proportion of an oblique force to move a body, is to that of the same force coming with a perpendicular direction, as the sine of the \(L\) of incidence, is to the radius. For the oblique force is compounded of two forces, the one \(\perp\), and the other \(\perp\) to the surface of the obstacle, which is only affected by the latter or sine of incidence, the oblique force being made radius.

10. Hence the forces of a fluid medium on a plane cutting the direction of its motion at different inclinations, are as the squares of the sines of the \(L\) of incidence. For the force of a particle is as the sine of incidence; and the number of particles, that strike in equal time, is as the sines of incidence; th. the forces of all the particles are as the squares of the sines of the \(L\) of incidence.

11. If the velocity of a medium be different, the forces on a plane cutting that medium with the same inclination, are as the squares of the velocity. For the force of each particle is as its velocity; and the number of particles, that strike in equal time, is as their velocities; th. the forces are as the squares of the velocity.

12. The force of the water upon the rudder of a ship in motion is as the square of the sine of the inclination of the rudder to the keel (by \(\theta\)); and the force of the rudder upon the keel is as the co-sine into the square of
of the Sine of that Inclination, i.e. as \( \frac{r^2 - s^2}{s} \times s^2 \);

which if a Maximum, then \( s^2 \times \frac{r^2 - s^2}{s^2} = 2s^2 \times \frac{r^2 - s^2}{s} = 0 \), Th. \( s = \sqrt{\frac{r^2}{2}} \) \( (\approx 54^\circ, 44^\circ, 08^\prime \text{ near}) \) = Sine of that Angle which the Rudder, in the most advantageous Position, Should make with the Keel.

13. And the Force \( F \) of a Fluid Medium upon a \( \Delta \) moving according to the Direction of its Base \( (b) \) is to the Force \( F \) on the Circumference, Parallelog. as the Sq. of the Perp. \( \gamma \) to the Sq. of the Hypotenuse \( (h) \) i.e. \( \Delta : F, \square : p^2 : b^2 \). And the Force, striking with the Velocity \( (u) \) upon the Surf. defcr. by \( p \), taken infinitely small, about an Axe, at the Distance \( y \), is as \( v \gamma \) (for \( y \gamma \) is as that Surf.) Th. \( b^2 : p^2 = v \gamma \cdot \frac{v \gamma}{b^2} \) the Force on the Surf. defcr. by \( b \), at the same time.

Th. if \( H, b \) be any two adjoyning Particles of a Curve,

\[ \frac{yp^3}{b^4} : \frac{yp^3}{b^4} + \frac{yp^3}{b^4} = \frac{yp^3}{b^4} = \frac{yp^3}{b^4} \] is the Force of the Fluid on the Surface generated by the Rotation of \( H + b \); which if a Minimum, then \( B, b \) being variable \( 2b^2yp^3 \times h^4 = -2b^2yp^3 \times h^4 \) or (bec.

\( B + b \) is constant, th. \( \frac{b^2}{h^4} = \frac{b^2}{h^4} \) or (sup. \( p = p \) \( xy \cdot \frac{x}{h^4} \) is an Invariable: The Property of the Curve that generates the Surface of a Solid, which moving in a Fluid Medium, according to the Direction of its Axis of Rotation, shall meet with less Resistance than any other Solid generated by a Curve described to that Axe, and passing thro’ the Extremity of the given Ordinate \( y \).

And drawing \( x \) to the Axe, and \( \tau \parallel \) to the Tangent, the \( \Delta \) made by \( x, \tau, \beta \) is Sim. to \( \Delta \) made by \( p, b, b \), th. \( 4x^3 \beta (4x^2 \cdot \frac{\tau}{x} \cdot \frac{\tau}{x} \cdot \frac{\tau}{x} \cdot \frac{\tau}{x}) \cdot \tau \) \( (x^3 \frac{\tau}{x} \cdot \frac{\tau}{x} \cdot \frac{\tau}{x} \cdot \frac{\tau}{x}) : \tau \) \( (\frac{\tau}{x} \cdot \frac{\tau}{x} \cdot \frac{\tau}{x} \cdot \frac{\tau}{x}) : y \).

14. Let the unequal Radii \( CL, Cl \) (Fig. in p. 289.) sustain the Weights \( P, W \), by the Cords \( LP, LW \); Required their Forces to move the Wheel: Sup. \( (x, w) CD, Cd \parallel \) to the Directions of \( P, W \), with \( CD \) defcr. a Circuml. cutting the Direction of \( W \) in \( g \); Then (since a Body hanging freely by any Point, makes that Point as heavy as if
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if it exist there) the Weight $W$ is as $g^2$, and its Force to turn the Wheel is as $g^2 D$. And if $W \cdot P = (g^2 \cdot g^2 D) \cdot gC$ or $CD : Ca$, i.e. if $P \cdot \omega = W \cdot \omega$, or when $P \cdot fP = W \cdot fP$. $W$, Then contrary Forces and Resistances will sustain one another; Th. if $P$ be greater than

\[
\omega \cdot \frac{\omega}{W} \cdot fP. W \cdot \frac{fP}{P} \cdot P \cdot W
\]

the Power will overcome the Resistance or Weight, with a Force equal to the Excess. Hence any given Weight or Resistance, may be moved or overpower'd by any given Force: Consequensly, the greatest conceivable Weights may be moved by the least conceivable Power. And,

15. In the Leaver, or Steel-yard; when $P \cdot \omega = W \cdot \omega$, the Power will sustain the Weight or the Weights will Equiperonderate. There, in the Balance, where $\omega = \omega$, the Weights must be equal.

This Property of the Leaver explains the Powers of Oars in Rowing; of Iron Crows in moving or lifting Weights; of the Hammer in drawing out Nails, &c. also of Pincers, Shears, &c. which consist of double Leavers bearing on a Common Fulcrum.

If two Fulciments $(x, \gamma)$ sustain a Leaver Horizontally, and $e$ be the Centre of grav. of the Burden; Then (with respect to the fixed Points $x, \gamma$) $p, x : W : \gamma : x \gamma, \& P, \gamma : W : x \gamma : x \gamma$. Th. $p, x : P, \gamma : x \gamma : x \gamma$; And $x, or \gamma$ will sustain $x \gamma : x \gamma$, or $x \gamma : x \gamma$ of $W$. Hence a Weight carried between two Men may be placed on the Leaver in any given Proportion; And so two Horses drawing any Weight, the Resistance may be divided according to the Strength of the Horses.

If two Fulciments $(x, \gamma)$ sustain a Leaver oblique to the Horizon, and $a, e$ its Intercensions with the $l$ to the Horizon, passing thro' the C. of Gravity of the Weight $(W)$ sustain'd Below, or Above that Leaver; Then $x, \gamma$ will bear

\[
\frac{a^2}{x} : x, \frac{a^2}{x} \text{ part}, \text{ or } \frac{e^2}{x} : x, \gamma \text{ part of } W. \text{ And}
\]

if the Centre of Grav. be Below, or Above the Leaver, the Superior, or Inferior Fulciment will bear so much the greater part of $W$, as the Leaver is more inclin'd.

If a Weight $(W)$ be placed on Three Beams $(ap, bp, \gamma)$ join'd in $p$, and supported by Three Props $a, \beta, \gamma$; suppose each Beam a Line, and produced thro' $p$, to meet the Sides of a $\Delta$ (whole angular Points are at $a, \beta, \gamma$, $a, b, c$; Then $a, \beta, \gamma$, will bear $pa : a, a, pb : \beta b, pc : \gamma c$ part of $W$. Hence,
Hence, in Building, where the Joysts or Girders are too short, they may be placed so as to sustain each other; and by the Combination of such, any Flooring, or the like, may be completed, and the Weight sustain’d at each Contignation may readily be computed.

Hence also, the Strength of Beams fix’d at one End is easily found: And that a Parabolic, or a Prismatic Parabolic Beam, has the same Resistance in all its parts; Theref. \( \frac{3}{2} \), or \( \frac{5}{3} \) of the Timber, will in this Case, do the same Service: Which in Civil and Naval Architecture is is of no small Importance.

16. In the Wheel; when \( \text{Power} \times \text{Rad. of the Wheel} = \text{Weight} \times \text{Rad. of the Windlace} \), the Power will sustain the Weight. Whence the Force of the Captain, the Crane, &c. as also of all Engines with Tooth’d Wheels, may be accounted for.

17. In the Pulley; (let \( n \) = Number of Falls or Cords at the Block hoist’d to the Weight) when \( P : W :: (S_p \times W : S_p : P :: )^t : n \), or when \( P = \frac{1}{n} W \), the Power will sustain the Weight.

18. In the Wedge, the Weight \( w \) lying on the two Oblique Planes \( at, \alpha \tau (\perp \lambda \tau) \) in Fig. p. 289.may be consider’d as a Wedge, where \( f, w \) on \( \alpha \tau : f, \text{Hammer} \) (according to the Line \( w \chi \)) on that Plane \( :: t \chi : w \chi \), and \( f, w \) on \( \alpha \tau : f, w \) on \( \alpha \tau : t \chi : w \). And when \( \text{Force} \times \text{Altitude of the Wedge} = \text{Resistance} \times \text{Thickness} \), the Power will be Equivalent to the Resistance. This accounts for the Force of Hatchets, Chisels, and all Edge’d or Pointed Instruments, as also Saws, Files, &c.

19. In the Screw, which may be taken as a Wedge impell’d by a Leever; When the Power that turns the Screw \( \times \text{Circumf.} \) defr. by it = Resistance of the Obstacle (to be press’d or rais’d) \( \times \text{Distance between the Threads of the Screw} \), then the Power will be Equivalent to the Resistance.

20. Let a Weight \( (w = P) \) be sustain’d partly by a Power at \( L \), and partly by the Inclin’d Plane \( ab \); draw \( w \chi \), and \( w \perp \text{Horizon} \), and Plane; then the descending Force of \( w \) may be express’d by \( w \chi \), which may be resolv’d into the Forces \( w, t, \chi : \text{Th.}, f, w \) descending along the Incl. Pl. \( ab : f, w \) along the Perp. Pl. as : : \( (t \chi : w \chi : s, \perp \lambda \chi : s, \perp \lambda \gamma : :) \) s, \perp \text{Inclination : c}, \perp \text{of Traction}.

21. Th. if a Power at \( L \) sustain a Weight \( (w) \) upon an Inclin’d Plane, by a Direction \( (\lambda w) \) which, passing thro'
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the C. of Grav. of w, meets the Plane; Then Power : Weight :: \( s \cdot L \) Inclination : \( cs \cdot L \) Tradition. If \( Lw \parallel ab \), Then (bec. \( \Delta wm \sim \Delta bsa \sim \Delta S a \), if \( SF \parallel ab \)) Power : Weight :: \( (sx : xw ::) as : ab :: sp :: s, \) Incl. : \( cs, \) Incl. And Power at \( L \), when \( Lw \parallel ab \) : Power at \( L \), when \( Lw \parallel sb :: sb : ab \).

22. If \( as, ab \), be two Planes, the one \( \perp \) the other Oblique to the Horizon, let \( sF \perp ab \); Then the Initial Forces, or Celerities of Descents, acquir'd in the same time, are :: \( (sa : aF ::) ab : sa, i.e. \) are reciprocally as the Lengths of the Planes.

23. And in equal Times, Sp. run in \( ab :: \) Sp. run in \( as :: \) (vel. in \( ab :: vel. in as :: as : ab :: aF :: as, \) i.e. \( The Weights \) shall descend thro' \( aF \), and as in equal Times. And (drawing \( \mu \parallel sF \)) the Spaces \( \mu s, aF \) shall be pass'd over in the same Time.

Th. If the Diameter of a Circle be \( \perp \) to the Horizon, a Body shall descend in the same Time thro' any Chord whatsoever concentric to that Diameter.

24. Times in \( ab :: as :: (Times in \( ab :: aF :: \) \( \therefore ab :: as, \) i.e. Times of Descents thro' Planes equally high are directly as the Lengths of the Planes. Th. (if \( \mu F \parallel sb_1 \)) Times in \( \mu F :: \) \( sF, \mu s. \)

25. And since \( v, \) in \( s :: v, \) in \( F :: as : aF :: \) \( \therefore ab :: aF :: as, \) i.e. The Velocities acquir'd in falling thro' Planes equally high are equal.

26. Or, let a Body descend by a Line (\( l \)) inclin'd to the Horizon, with a Velocity increasing as the Time (1.) Sup. \( bh \) the Sine and Co-Sine of the Inclination, \( v \) the last Velocity; Then (\( l \) and \( c \) being variable) \( l = cc \div l \), but \( v \) is as \( \sqrt{b}, \) th. \( t, \) in \( l \) is as \( cc \div \sqrt{b} \times l \) or \( cc \div \sqrt{b} \times \sqrt{b}^{\frac{1}{2}} + c^{\frac{1}{2}} \), th. \( t, \) in \( l \) is as \( \sqrt{b^{\frac{1}{2}} + c^{\frac{1}{2}}} \div \sqrt{b} \) or \( l \div \sqrt{b}, \) th. \( t^2 \) is \( l^{2} \div b, \) and \( HT^2 L^2 = b h^2 L^2, \) also (bec. \( vt = l \)) \( HTlv = bhLV. \) Hence,

1. \( T :: :: L \times \sqrt{b} ; L \times \sqrt{H} ; \) And \( V :: :: 1HT ; 1bt. \)

2. If \( L :: H :: b, \) Then \( T :: :: \sqrt{L} ; \sqrt{L} : V :: v. \)

3. If \( H = b, \) Then \( T :: :: L :: l \); And \( V = v. \)

4. If
4. If \( L = l \), Then \( T : t :: \sqrt{b} : \sqrt{H} \), And \( V : v :: s : T \).

5. If \( L^2 = l^2 H \), Then \( T = t \), And \( V : v :: L : l \). Th. the Times of Descents thro' the Chords \((L,l)\) of Arcs (whose vers'd Sines are \( H, h \)) are equal. For, in a Circle \( L^2 = l^2 = H : h \) (by 65.3).

27. If a Body descends thro' how many ever contiguous Planes, however inclin'd, it shall acquire the same Velocity in the lowest Point, as if it had descended by the Perpendicular. Th. a Body descending by the Circumf. of a Circle, or any Curve Line, shall acquire in the Lowest Point, that Velocity which it would get by a Perp. fall from the same Height. And, bec. a Body thrown upwards with that Velocity which it got left by a Perp. fall, does ascend to the same Height, a Body if carried upwards (with the Velocity acquired in the Lowest Point, in descending by any Curved Surface) by the same, or any other Surface, however inclin'd, shall ascend to the same Height from whence it came, and have, in Points equally high, the same Velocity.

28. The Velocities of a Pendulous Body (as \( s \)) describing different Arcs \((s, s')\) are in the lowest Point \((s)\) as the Chords of those Arcs. For (drawing \( sF, s'F \perp as\), \( v \), in \( s \) are as \( \sqrt{s} : \sqrt{s} :: \sqrt{v} : \sqrt{v} :: s : s'F \).

29. And (since the Direction of the Impetus of a Body descending by a Curve Line is Horizontal in the lowest Point) if a Pendulous Body be struck in the lowest Point \((s)\) by an Horizontal Force equal to that in falling thro', it shall ascend from thence to the same Height. Since the smallest Arcs coincide with their Chords, which in the lowest Point are Isoronal, Th. the smallest unequal Oscillations are performed in the same Time. But greater Arcs are not as their Chords, Th. Longer Pendulums, thus describe fewer Degrees, err less; And greater Arcs take up a little more time than shorter ones do.

30. Let MAC, mac be several contiguous Planes, equally inclin'd to one another, and to the Horizon, and of Proportional Heights; Then the Times in which a Body runs thro' those Planes, shall be in a Subduplicate Ratio of their Altitudes MB, mb. For \( t, MA : t, ma :: \sqrt{MA} : \sqrt{ma} :: \sqrt{FA} : \sqrt{fa} :: \sqrt{FC} : \sqrt{fc} :: \sqrt{FA} : \sqrt{fa} : t, AC : t, ac :: t, MA + t, AC, &c. : t, ma + t, ac, &c. :: \sqrt{MB} : \sqrt{mb} \).
Th. in two Pendulums describing like Arcs (if $L, l = \text{Lengths}, T, t = \text{Times}, N, n = \text{Number of Vibrations in the same Time}) T : t :: \sqrt{L} : \sqrt{l}; \text{ And (since } NT = nt) N : n :: L : l. \text{ Th. if } L = 39,125 \text{ Inche, and } N = 60 \text{ Vibrations in } \frac{1}{11} \text{ of Time; Req'd. } l, \text{ so as to make } n \text{ or upwards } 100 \text{ Vibrations in that Time. Then } l = LN^2 \div n^2 = 39,125 \times 60^2 \div 100^2. \text{ Also if } l = 20 \text{ Inches, and } n \text{ the Vibrations in } \frac{1}{11} \text{ be require'd, Then } n = N^2 \div l = 60^2 \times 39,125 \div 20^2.

31. The Times, in which a Weight descends from any Point (o) of a Cycloid, are equal among themselves: And have to the Time of a Perpendicular fall thro' (as) the Axe of the Cycloid, the Ratio of $\frac{1}{2}$ the Circumf. of a Circle to the Diameter.

For, let $\omega \beta \gamma \alpha$, and any Rt. Line $pm || \omega \beta$, cutting the $\frac{1}{2}$ Circumf. or $sa, sb$ in $e, r$; let $pe$ and $re$ touch the Cycloid and Circle in $p$ and $r$, draw $e\mu || pm$, and suppose 'em infinitely near. Then will $re$, and $pe$ coincide with their respective Arcs; But $pe : mu (== gs : sm :: \alpha (sa) : s\phi (by \theta 5.2)) :: \sqrt{a} : \sqrt{sm}$, th. $pe = \frac{mu \times \sqrt{a}}{\sqrt{sm}}$, and vel. in $p$ or $m$ is as $\sqrt{m\beta}$, th. $t$, in $pe$ is $\left(\frac{pe}{\sqrt{m\beta}} = \frac{mu \times \sqrt{a}}{\sqrt{sm\beta}}\right)$

\[
\frac{mu}{mr} \sqrt{a} = \frac{re}{cr} \sqrt{a} = \frac{re}{t} \times \sqrt{a} = \frac{re}{d} \times \frac{a}{\sqrt{a}} = \frac{re}{d} \times t,
\]
in $a$. Th. $t$, in all the $ps's (i.e. in the Cycloidal Arc = $s$)

is = to all the $re's$, or $\frac{t}{d} \div d \times t$, in $a$, And $t$, in the Cycloid $\times d = t$, in the Axe $\times \frac{1}{2}$. 

32. Hence
32. Hence, \( c : d \cdot t \), of 1 Oscillation, of Decem in \( \frac{1}{2} \) the Length (l) of the Pendulum. For \( l = \) twice the Axe (by 72.4).

Put \( \lambda \) for the Length of a Pendulum, that measures 1/11 of Time in each Oscillation; Then \( c : d \cdot u (\tau, 11') : 2v \cdot e \) = Time (\( \tau \)) in \( \frac{1}{2} \lambda \); And (\( \tau^2 : v^2 : : d^2 : c^2 : \frac{1}{2} \lambda : b \)) = Height fallen from in 11/11 of Time. But 'tis found, by means of Clocks adjusted by the Heavens, that \( \lambda = 39,125 \) Inches London Measure, Th. \( b = \frac{1}{2} \lambda \times c^2 \div 12 d^2 \) = 16,089.5 Feet, or 16 F. and about 1 Inch.

The Reflactance of the Medium does somewhat vary the Time of Descent in a Cycloid; But in a Medium that does not Reflact, the shorter Oscillations in a Cycloid are nearly Isocronal. And the greater the Funipendulous Body is, the less does the Medium Reflact it; But then the greater is the Distance of the Centre of Gravity from the Centre of Oscillation (i.e. the Point whose Distance from the C. of Suspension is the Length of a Simple Pendulum, whose Vibrations shall be Isocronal to those of the given Magnitude, or that Point wherein all the Figure is supposed to be contracted, with the Forces, while it vibrates.) A Right Line parallel to the Horizon, about which the Oscillation is made, is called the Axis of Oscillation. And every right Line, or Plane passing thro' the Centre of Gravity is an Axis, or Plane of Equilibrium.

33. Given two Weights \((W, w)\) and the Distance \((Dd)\) between their Centres of Gravity \(D, d\): Req'd. (g) their Common Centre of Gravity, or that Points in which all their Forces unite, or where, if they be jointly suspended, they'll produce the same Effect as they did separately. Let \((W : w : : Dg : i. e.) W : w : W : DD : Dg, or W + w + w : Dd : Dg\). Th. if there be any given Weights A, B, C, &c. and the Distance between their Centres of Gravity; Then (g) the Common Centre of Gravity of 'em all is easily found: For let \(a\) be the Centre com. to A and B, also \(b\) the Centre of A + B and C, &c. And if any Weight A, B, C, &c. be apply'd to any Points of a Line suspended at a given Point, and distant from it by \(a, b, c, \&c.\) Then will (d) the Distance of their Common Centre of Gravity (g) from the Point of Suspension be \( = \)

\[
\frac{W + w + w + a + b + c + \&c.}{W + w + w + a + b + c + \&c.} = \frac{A + B + C + \&c.}{M + m + m + \&c.}
\]

Also, if there be given several Weights A, B, C,
A, B, C, &c. (in the same or different Planes) whose Distance from the Axis of Oscillation call \(a, b, c, &c\), and the Distance of their Common Centre of Gravity from the same is \(d\); Then the Distance (\(d'\)) of the Centre of Oscillation from the Point of Suspension (or the Length of a Simple Pendulum that shall move as fast as a Pendulum composed of all the Weights) is equal to

\[ \frac{Ax^2 + Bx^2 + Cx^2}{&c.} = \frac{A + B + C}{&c.} \times d, \]

that is,

\[ \frac{F + g + f, &c.}{M + m + m, &c.} \]. Th. if the Weights are equal, and their Number \(n\), then \(d = a^2 + b^2 + c^2, &c. \div nd\).

34. In Quantities that are Suspended to, or do Oscillate about an Axis (A); Let \(d, d'\) be the Distance of the Centre of Gravity, or of Oscillation from (A) the Axis of Suspension, or of Oscillation: x the Abcissâ of a Curve whose Ordinates (y) are parallel among themselves, and right to the Diameter; \(\lambda\) the Length of the Curve; \(\alpha\) the Area adjacent to the Abcissâ or its Parallel, and Flowing 1 to A; \(\chi\) the Surface of a Solid generated by the Rotation of a Plane adjacent to the Abcissâ or its Parallel, about any Side of the Curve's circumference. Pgr. and Flowing 1 to A; \(\lambda, \alpha, \chi\), or \(\lambda, \alpha, \chi\); and \(\phi\) the Fluent of any Fluxion: Then,

1. Where \(A\) is a Tangent to the Verses of the Curve; \(d = \phi, mx \div m\); \(d' = \phi, mx \div \phi, mx\).

2. Where \(A\) is the Base of the Curve; \(d = pm - \phi, mx \div m\); \(d' = \phi, p - x^2 \div m = \phi, p - x \div m\).

3. Where \(A\) is the Abcissâ; \(d = \phi, my \div m\); \(d' = \phi, my = \phi, my\).

4. Where \(A\) is parallel to the Abcissâ; \(d = pm - \phi, my \div m\); \(d' = \phi, p - y^2 \div m = \phi, p - y \div m\). Examples in Case I.

If \(y = x^2\), then \(\alpha = (x^2)x^2\), th. \(\alpha = x^{n+1} \div n + 1\).

Th. \(d = (\phi, ax^2) \div x \div n + 2\); And \(d' = (\phi, ax^2) \div (n + 2) \times x \div n + 3\).

Also,
Also, bec. \( \dot{x} = \frac{x^{2n+1}}{n+1} \) \( r, \) & \( \dot{x} = \frac{x^{2n+1}}{2n+1} \) \( r 2n+1 \)

Th. \( d = \frac{x^{2n+1}}{2n+1} \) ; And \( d = \frac{x^{2n+1}}{2n+1} \)

**Note.** A Surface, or Solid generated by the Uniform Rotation of a Line, or Surface about an Axe, is equal to a Surface, or Solid whose Base is that given Line, or Surface, and whose Altitude is \( \frac{c}{\text{Periphery of its Centre of Gravity}} \).

In a Cylinder (whose Height \( = h \), Rad. of its Base \( = r \)) suspended by \( PE \) the Extremity of the Axe, The Distance \( (gc) \) from \( g \) (the Centre of Grav.) to \( c \) (that of Oscillation) is \( \frac{1}{2} Pg + r^2 \). And in a very small Rod \( (PE) \) suspended by the End \( P \), \( gc = \frac{1}{2} g E \). But if \( PE \) be taken as the Diameter of a Sphere suspended by \( P \), then \( cg = \frac{1}{2} Pg \). And if a Sphere be suspended from another Point \( (\omega) \) then \( cg = \frac{1}{2} Pg \times \frac{1}{2} g \), which gives the Centre of Oscillation in Order to add the Pendulum of a Clock. So that if \( \omega g \) remain the same, then \( cg \) will be as \( Pg \); Th. if \( \omega g = 39,125 \) Inches, and \( Pg = 1 \), then \( cg = \frac{1}{2} Pg \) nearly; Also if \( \omega g = 39,125 \), and \( Pg = 3 \), then \( cg = \frac{1}{2} Pg \) nearly.

If \( \omega g \) the Distance between the Centres of Suspendion and Gravity be the same, then the Greater the Sphere is, the Slower will it Oscillate; But if neither the Bigness of the Sphere, nor \( \omega g \) be alter'd, then, in Resisting, or Non-resisting Mediums, the Lighter the Sphere is, the slower, or quicker will it Oscillate. And without having Respect to the Centre of Oscillation, the Length of a Pendulum for determining any Time, or for any Parallel of Latitude, cannot be accurately found.

35. Suppose a Body in the 1st Moment of Time to run throu' \( ab \), theref. the same Velocity continuing, it shall run in the 2d Moment throu' \( bc \); but in \( b \), sup. the Centripetal Force acting, draw \( \gamma c \parallel bc \), and = to the Length thro' which the Body is carried towards \( C \) in a Moment, then \( bc \) is run thro' in the 2d Moment; and the \( \Delta:Cb = \Delta:Cb = \Delta:baC \).

Likewise in the next Moment, will the Body describe...
scribe \( \triangle \), making the \( \Delta \triangle \delta \) \( \equiv \triangle \gamma \delta \), and so on; whence, equal Areas are describ'd in equal Times. Let as well the Moments of Times, as the right Lines \( ab, \), \( bc, \) &c. be conceiv'd infinitely small, then the Polygon \( abcd \) will become a Curve Figure, from whose Tangent the Body is perpetually retracted by the Vis-Centripeta; And the Areas describ'd by the Radii drawn from the Centre (C) to the Body shall be Proportional to the Times of Description.

And if a Body move in a Curve Line, and by Rays drawn to the same Point describ' the Areas Proportional to the Times, it is retracted from the Tangent to this Curve by a Centripetal Force tending to this Point.

36. Let \( C \) be the Centrifugal or Centripetal Force, \( T \) the Periodical Time, \( V \) the Velocity; Sup. \( AB \) a Tangent, \( BH \) a Secant, then \( BE = AB^2 \div BH \) (by 64.19) But if \( AB \) be infinitely small, then will \( BE = AE^2 \div EH \); Th. (if \( AE, ae \) be Areas describ'd by two movable Bodies in the same Time; Then \( C : : (AE^2 \div 2R; ae^2 \div 2r : : V^p : u^2 R : : R^t : r^t ; Th. \) if \( T^1 : t^1 : R^1 : r^1 ; \) then \( CR = cr \); And \( v = u \). If \( T^2 : t^2 : R^2 : r^2 ; \) then \( CR^2 = cr^2 \); And \( v^p \: v^r = u^p \: u^r \).

'Tis observ'd, That the Primary Planets do describe equal Areas under the Sun in equal Times; And that the Squares of their Periodic Times are as the Cubes of their Distances from it; Th. 'tis the Sun that is the Centre of the Planetary System.

It follows also, that the Force of Gravity of the Heavenly Bodies is Reciprocally as the Squares of their Distances from the Centre. And \( V : v = R : \sqrt{R} : \sqrt{R} : R, \) the nearer the Centre, the greater the Velocity.

If 2 Bodies describe equal Areas (\( CAE, CE \)) in equal Times, or if \( CA \times AB = ce \times ac, \) i.e. \( VR = ur, \) and \( V^2 R^2 = u^2 r^2 \) Then \( C : : (V^2 \div R : \sqrt{v^2} \div r : \sqrt{r^3} \div R : R \div \sqrt{r} : \sqrt{r} : R^3 \).

37. The \( \lambda, \alpha, \) (which, a Body describ. about the Centre of Attraction) are as \( R^2 \) to \( R^3 \). For \( V, v = RA, RA : \sqrt{r}, \) th. \( A : \alpha, (\sqrt{VR} : VR, \) (bec. if \( T = t, \) then \( V : v = t : R, \sqrt{r} : R^3 \).

38. If \( v^p = V \) Velocity acquired in the Time \( t, T, \) \( s = \) Space run, supposing, \( T : T^p : v^p : V^p; \) \( s = \) Space run by an Equable Motion in the Time \( T, \) with Velocity \( V, \) (let \( m \)
Synopsis Part 2.

\[ x = \frac{1}{2} \frac{x}{v} = x \] Then \( s = T v^x = \frac{x}{v} T v^{x-1} \)

\[ \frac{V^x}{v} \] but \( s = \frac{x}{v} T v^x = \frac{x}{v} V^x \)

\[ T v^{x-1} = \frac{x}{v^x} \]

(putting \( V \) for \( v \)) \( m = \frac{x}{v^x} \cdot T V \);

And \( S = V T \), Th. \( s = S : m = m + \frac{x}{v^x} \).

Or if \( T = s = V \)

\( v \), then \( S = m \).

Th. if \( S \) and \( m \), = Space run by an Equable, and an Equably Accelerated Motion in the Time \( t \), and \( v \), Then

\[ s = v^x : 2 : S : \frac{x}{v^x}, \quad \text{and} \quad S : \frac{x}{v^x} = 2 : v^x. \]

39. Let the Arc \( a \) be descr. by a Body revolving in a Circle with an uniform Motion in the Time \( t \); and the Rt. Line \( b \) descr. by its Deceit with Accelerated Motion in the Time \( v t \); then (bec. \( a : b =: 2 : v^x \), \( a = 2 b : v^x \),

but the Centrifugal Force \( c \) in the time \( t \) is \( a \) = 2 \( r \) (by \( 36. \)) = \( 4 b^2 \cdot 2 r v^4 \), Therefore the Space run by the Centrifugal Force \( c \) in the Time \( t \) is run by (Gravity) \( g \) in Time \( v t \) (\( 2 b^2 \cdot 2 r v^4 : b \) = \( 2 b : v^4 \).

Th. if a Body (whole Centrifugal Force is \( c \), Gravitating Force \( g \)) moves uniformly in a Periphery \( (p) \) whose Rad. is \( r \), with a Velocity equal to that acquired by falling the Height \( b \); Then shall \( c : \frac{g}{v} = 2 b : r \), Th. if \( b = \frac{1}{2} g \).

40. If any two Pendulums, carried with a Conic Motion, descr. Peripheries \( (P, p) \) whose Radii are \( R, r \); The Times \( T \) of Description shall be as the Square Roots of the Altitudes \( (A, a) \) of the Cones. For \( g : e = a : r \), th. \( c = \frac{1}{2} \frac{g}{v} \cdot \frac{2 b}{r} \cdot r \), and \( b = (r^2 - 2 a) v^2 \), th. \( v = r \cdot \frac{1}{v^{2 a}} \), but

\[ \frac{p}{v} \cdot \frac{2 a}{v} = \frac{r}{v} \cdot \frac{2 a}{v} = \text{Time} \text{ in } \frac{p}{v} \text{. And} \frac{p}{v} \cdot \frac{2 a}{v} = \frac{r}{v} \cdot \frac{2 a}{v} \]

P\( \times \frac{2 a}{v} = R \), \( v \) = \( \sqrt{A} \cdot T \). Th. if \( A = e \), then \( T = t \).

41. In a Pendulum carried by a Conic Motion; Time in descr. the least Periphery; Time in falling an Alt. = twice the Length \( (L) \); Circums. : Diameter. For \( v \), in \( 2A \), is \( \sqrt{2A} \), and Time is \( 4A = \sqrt{2A} \) or \( 2 \sqrt{2A} \), And \( 2 \sqrt{2A} \):

\[ \frac{P \times 2 A}{v} = R \cdot P \times \frac{2 A}{v} = \text{Diam.} \cdot \text{Circums.} \]

But (bec. \( R \) is sup. infinitely small) \( L = A \), Th. \&c. Th. the Time in describing the smallest Circul is = to the Time of the two smallest Oscillations of the Pendulum.

42. If
42. If \( p \sqrt{\frac{2a}{r}} \) Time in descr. \( p = (2Vl) \) Time in falling Alt. = \( l \); Then (bec. \( 4l = 2p^2a \div r^2 \)) \( a : l :: 2p^2 : p^2 \), or s. **Incl. Rad. :: Inscr. Sq. :: Sq. Circumf.**

43. If a Pendulum descends thro' a Quadrantal Arc, the Vel. in the lowest Point is \( \frac{2a}{r} \) to that acquired in falling thro' \( r \), th. \( c \frac{2ag}{r} \) = \( 2g \), Th. the Force in the lowest Point is \( (2g^2 + g) \frac{3g}{2} \).

44. The Spaces, which a Body impell'd by some regular Force describes, are in the very Beginning of the Motion, as the Squares of the Times. For if the Times be express'd by \( ve \), \( \int \) (Fig. Art 69.7) and the Celerities in those Times, by \( ep \), \( \tau \), then the Spaces shall be as the Areas \( vpe \), \( vns \), which are as \( ve^2 \) to \( \int \) (by 69.7.)

45. If a Body in revolving about a Centre \( (s) \) describes a Curve Line \( vns \); Let \( ps \) be a Tangent, \( s \parallel \) and \( s \perp ps \); Now time \( (T) \) is as \( \Delta s \tau p \) (by 35) or as \( \pi p \times sp \), and if \( T \) be given, the Linolea \( \tau \) is as \( C \) (the Centripetal Force), but if \( C \) be given, \( \tau \) is as \( T \), or as \( \tau p \times sp \), theref. universally, \( \tau \) is as \( C \times \tau p \times sp \), th. 

\[ c = \pi u \times sp. \]

Example. If a Body revolve in an Ellipse; to find the Law of the Centripetal Force tending to the Focus. Draw the Diameter \( ps \), and \( ab \) Conjugate to it, cutting \( ps \) in \( a \); Let \( pm \perp ab \), and \( px \perp ps \); cutting \( ps \), \( pc \), \( pm \) in \( r, n, x \); Then \( ap \) = \( cv \) (by 68.4) and \( cb \times pm = cd \times cv \) (by 68.28.) also (bec. \( \pi x = \pi t = 1 \div 0 \) \( \pi r = \pi x \), and \( \tau n = \tau p \); But \( \tau u^2 = \pi r^2 \) (\( \pi m^2 = \pi a^2 \); \( \pi p^2 = \pi c^2 \); \( \pi u^2 = \pi b^2 \); \( \pi r^2 = \pi n^2 \); \( \pi p^2 = \pi c^2 \); \( \pi c^2 = \pi b^2 \), th. \( \pi \tau u^2 = \pi c^2 \) (\( \pi p^2 = \pi cp \); \( \pi p = \pi (cp + q) \); \( \pi c = \pi cp \); \( \pi cp = \pi cp \frac{1}{2} \pi ab \)); th. 

\[ \frac{1}{2} cu, \text{th.} \frac{1}{2} u = \frac{1}{2} cv \div cd^2, \text{th.} C \text{ or } \frac{1}{2} \pi u \times sp^2 \]

\( \pi = \frac{1}{2} cu \div cd^2 \times sp^2 \) (since \( \frac{1}{2} cu \) and \( cd^2 \) are Standing Quantities) Thus is, the Centripetal Force is reciprocall as the Square of the Distance.

And by a like reasoning, 'tis readily found, that if a Body move in an Hyperbola, or Parabola, the Centripetal Force
Force tending to the Focuse, shall be reciprocally as the Square of the Distance. But if the Centripetal Force should tend to the Centre of the Ellipse or Hyperbola (which is no where found) it would be directly as the Distance.

Hence, if a Body move in a Rt. Line from any place, with any Velocity, and with a Centripetal Force that is reciprocally as the Square of the Distance, it shall move in some of the Conic Sections, whose Focuse is in the Centre of Forces.

46. Let $A, a$ be the Areas describ'd in different Orbs or Ellipses $E, e$, whose Transverse Axes are $D, d$, Parameters $P, p$, let $T, t$ be the Periodical Times. Then if several Bodies revolve about a Common Centre, and that $C$ is as $\frac{1}{2} ps^2$, Then (bec. $P = \frac{cd^2}{2} + \frac{1}{2} tv$, by 68.5) $\nu u^2 = \frac{tv}{x}$ and in a given Time, $tv$ is as $C$ (by 45) or as $\frac{1}{2} sp^2$, th. $P$ is as $\left(\nu u^2 \cdot sp^2 = A^2\right)$. That is, $P : p :: A^2 : a^2$, Or the Parameters of the Orbs shall be as the Squares of the Areas describ'd in the same time.

47. Th. if $f = t$, $A$ is as $\sqrt{V} P$, but if $P = p$, $A$ is as $T$, Th. universally, $A : a :: \sqrt{V} P : \sqrt{V} p$, but $A : a :: \sqrt{V} P :: \sqrt{V} D^3 : \sqrt{V} px \cdot \sqrt{V} d^3$ (by 68.26) th. $T^3 = t^2 :: D^3 : d^3$. That is, the Squares of the Periodical Times are as the Cubes of the Transverse Axes, or mean Distances from the Centre of Attraction.

48. The same things being supposed (as in 45.) draw $s = (\vec{j}) \perp pt$ (the Tangent,) Then the Velocity ($V$) of the Body is as the little Arc $pt$ describ'd in a given Time, i.e. $as pt$ or $\nu t$, but $\nu \cdot \nu u : sp :: sp : st$, th. $pt$ is as $\nu u \cdot sp \div st$, i.e. $V$ is as $(A \cdot s) \sqrt{P} \cdot \sqrt{d}$ or $\sqrt{P} \cdot \sqrt{d}$.

83. Of the Motion of Projects.

1. A Body being Projected by any Force moves in the Curve of a Parabola, unless so far as the Resistance of the Medium binds it. Thus, if $HK$ be $\perp$ to the Horizon, and any Rt. Lines $\mu \nu, mV$ be such, as that in the Time the Body by its Projectile Motion arrive to $\mu, \omega$, it may fall, by its Descending Motion, the Length $\mu \nu, mV$; Then (by 82.1.) $\mu \nu : mV :: H \mu^2 : Em^2$, or $HK :: HK :: ku^2 :: KY^2$, which is the Property of the Parabola.

2. The Horizontal Distances ($H, h$) of Projects, made with the same Velocities, at several Elevations ($E, e$) are as
as the Sines of the double \textit{Ls} of Elevations. For (if \( s = \text{Sine}, \varphi = \cos \text{Sine} \times (RL) = bs = s, \) and \( y(\text{HL}) = br = s \)

\[
\frac{ps}{r} = \frac{b^2r^2}{s^2}, \text{th. } b = \left( \frac{ps}{r} \right) = \frac{b^2r^2}{s^2} \times \frac{2s}{r} = \left( \text{bec.} \right)
\]

\[
2s = \frac{s}{r} = s, \sqrt{15.6s}, s, 2L \times \frac{s}{r}, \text{Th. } H : b :: s, 2LE : s, 2Le.
\]

2. Hence, when \( le = 45^\circ \), the Sine of its double is the greatest Sine; but the Ranges are as the Sines of the double \( Ls \), Th. the greatest Random is as \( 45^\circ \) Elevation. And bel. at \( 45^\circ \) Elevat. \( s, 2L = r \), th. \( b \) (or \( g \)) = \( \frac{2s}{r} \), i.e. The greatest Random is equal to \( \frac{1}{2} \) the Parameter. Also, the Ranges equally distant above and below \( 45^\circ \) are equal.

3. The Altitudes (\( A, a \)) of Projectiles, made with the same Velocity, at several Elevations (\( E, e \)) are as the Vered Sines of the double \( Ls \) of Elevations. For \( s : s : (b) \)

\[
\frac{ps}{r} = \frac{s}{r}, \text{And } a \left( \text{in } V = Vm \right) = \frac{s}{r} = \frac{2s}{r} = \frac{2s}{r} \times \frac{1}{4r} \]

\[
\frac{1}{4r} = \left( \text{bec.} \right) \frac{2s}{r} = s, \sqrt{15.6s}, s, 2L \times \frac{s}{r} = \frac{p}{4r} = \left( = s, 2L \times \frac{s}{r} \right)
\]

Th. \( A : a :: s, 2LE : s, 2Le. \) And since \( 2r = \text{greatest vers'ed Sine}, \) th. \( r : 2r :: \frac{1}{2}r : \frac{1}{2}r = \text{greatest Altitude} = \frac{1}{4} \) the greatest Random.

4. The times (\( T, t \)) of the Flights of a Projectile thrown with the same Velocity at different Elevations (\( E, e \)) are as the Sines of the Elevations. For let \( y(\text{HL}) \) represent the Time, then \( p : p :: \frac{r^2}{r^2} = y, \) th. \( r : p :: s : s ; Y, \) i.e. \( T : t :: s, LE : s, Le. \)

5. Given \( b \) the Horizontal Distance of an Object, \( c \) the \( L \) of Elevation, \( n \) the Ascend or Descend. \( * \) Or. \( s, e : b :: s, e : x = \pm n, \) and \( b + x = \pm n = y. \)

Or \( r : s, e : b :: y, \) and \( y = x = p = 2g. \)

2. Since \( x = \text{Fall in the Time } y, \) th. 192 Inches : \( x :: \)

\[
\boxed{11/2 = \frac{x}{11/2} \times x \quad 11'''' = \text{Sq. of the Line in } x, \text{Th. Velocity (} S, T )} \]

\[
\frac{y}{\sqrt{11/2}} x = \frac{v}{\sqrt{11/2}} x = \frac{v}{\sqrt{11/2}} x = 193 v.
\]

6. Since the \textit{Force} or \textit{Velocity} of a Project, describing a Curve Line, is compounded of the Uniform Velocity in \( y, \) and of that Uniformly Accelerated in \( x; \) which will carry it but \( \frac{1}{2} \) the Space that it would run uniform-
ly with the least acquire'd Velocity in the same Time, (by 33. 82.) Therefore, the Velocity in any Point \((p)\) of the Curve is to the Velocity impress'd in \(H\), as the Tangents to the Ordinates; or as the Lengths of Tangents, to the Points \(P\), and \(H\), intercepted between the Diameters to those Points; or as the Secants of the \(Ls\) made by those Tangents produce'd, and the Horizontal Line. Th. the Least Force is at the Vertex of the Curve, or at the Height of \(\frac{1}{4}p\). And at equal Distances from the Vertex, the Forces are equal. Also, Least Force : Impress'd Force at \(H \div r \div \sqrt{c}\); \(Lc\).

7. Given, \(Lc, g \left(\frac{1}{4}p\right), n, \) and \(Lc\); Required \(b\).

Bec. \(ys \div r = x \div n\), th. \(x = ys \div rm \div r = y^2 \div p\), Th. \(y^2 = spy \div prn\); And \(r \div y \div cs, e \div b\).

8. Given, \(b, m, g \left(\frac{1}{4}p\right);\) Required \(E, e\), (whole Tangent), call \(t\). Since, \(bs \div r \div (b) = x \div n\), th. \(b \div n = x \div 1\) but \(b \div n \times p \div (=xp = y^2) = b^2 \div b^2\), and \(b^2 = bp \div np - b^2\). Therefore, \(t \div r = \frac{p \div 2b}{f} \div \left(\frac{1}{6}ight)\)

\[p^4 + 4pu \div 4b^2 - 1 \div \sqrt{r \times \frac{1}{4}p \div \sqrt{\frac{1}{4}p^2 + pn - b^2}}.\]

Th. \(b^2 \div \frac{1}{4}p \div 2n \div p^2 \div 4pu \div 4b^2\) to \(a^2\), whose Tangent is \(f\), And \(f = \frac{1}{4}p \div b = t, E, e, f\\text{ought.}\)

1. Th. if \(f > \frac{1}{4}p \div b\), or if \(pn > b^2\), then, in Descents, the Direction of the lower Elevation will be below the Horizon. But if \(pn = b^2\), the Direction will be Horizontal, and \(pr \div b = \text{Tangens of the upper Elevation.} \) And if \(b^2 + pn > \frac{1}{4}p^2\), the Object (in Ascents, or Descents) is beyond the reach of the Project thrown with that Velocity.

2. If \(\frac{1}{4}p^2 \div b^2 \div pn\), in Ascents, or Descents, there can be but one Elevation (whole Tangent is \(pr \div 2b\)) that will reach the Object. Th. \(n = \frac{1}{4}p - b^2 \div p\), or \(b^2 \div p - \frac{1}{4}p\), which determines the utmost Height of any jet upon each given Horizontal Distance. And \(ip = \sqrt{b^2 + n^2} \div n = \text{Hori. Range at 45° of a Project,} \) thrown with the least Velocity capable to reach the
the Object; or the Charge requisite (in Ascents, or Descents) to hit a Mark is that which at 45° Elevation would throw the Shot the Distance of $\sqrt{b^2 + n^2} \pm n$ upon the Plane of the Horizon.

3. The $b: r:: b^2 + n^2 : n$ (or $t \pm n$, or $t \pm t$, Incl.) is the Tangent of an Elevation proper for a Gun so charg'd, to strike the Object with the greatest Certainty and Advantage: But $t = ct$, or $t \angle 90°$ - Incl. (by 16. 65.) Thus the utmost Range, on an inclined Plane, is made, when the Axis of the Piece makes $\angle 45°$ with the Perp. and Object; so that there needs no more, than to let the Axis of a Piece, duly charg'd, bisect the $L(a)$ between the Perpendicular and Object.

4. The Geometrical Construction of the last Problem may be thus. Let $OB(\perp OH) = \frac{1}{3} p$, and $BC(1|OH) = HP$.

From $c$, with $BP (\frac{1}{3} p \mp n)$, describe an Arc, which (if the thing be possible) will cut $OB$ in $l, l$; then will $HI$ be the Lines of Direction sought. For (if $BP = w$, $Bl = u$, $HP = \tau$) $w^2 - \tau^2 = u^2 = (\text{bec. } u = w - x) w^2 - 2wx + x^2$; th. $\tau^2 + x^2 = (2wx)px = 2nx; \text{ and } y^2 = (\tau^2 + x^2, Q \alpha 2$.}
$300 \quad \text{Synopsis} \quad \text{Part 2.}$

$\pm 2nx = px \mp 2nx \pm 2nx = px$; i.e. $Hl^2 = pxPq$.

th. the Curve will pass thro' P. Or bec. $u = (m^2 - r^2)^{1/2}$

$= \frac{1}{2} \left( \frac{p + n}{b} + \frac{n}{b} \right)^2 = \frac{1}{2} p^2 \pm pn - b^2$ therefore $O = \frac{1}{2} p \pm u$, and $b : r = \frac{1}{2} p \pm u$: $(t, E, e)$

$r \times \frac{1}{2} p = \frac{1}{2} p^2 \pm u - b^2 = b$. Otherwise, Let CH

$AP, AH (AH' = \frac{1}{2} p)$, and AC \parallel HO; From C, with

CH, describe an Arc cutting (if possible) O in 1, 1; Then

will HL be the Lines of Direction. For since $AC = \frac{1}{2} pn - b (= c)$, therefor $CB = c^2 \pm pn + b^3$, and $Cl^2$

$(CH^2) = c^2 + \frac{1}{2} p^2$, th. $BI = \frac{1}{2} p^2 \pm bn - b^2$ at

And $b : r = \frac{1}{2} p \pm u$: $(t, E, e)$ $r \times \frac{1}{2} p \pm u - b$. Hence the 5th, 7th, and 8th Proposition may be Arithmetically Solv'd thus,

1. $(\frac{1}{2} E + e = 90^\circ - \frac{1}{2} \alpha$, th. $90^\circ - \frac{1}{2} \alpha \cos e = \frac{1}{2} E - e$

and $v, \alpha (AZ) - v, E - e (BZ) = q (AB)$, Then $q : s, \alpha (AH)$,

$= b : \frac{1}{2} p = g$.

2. Having found $q$, then $s, \alpha : q = \frac{1}{2} p : b$.

3. Let $\frac{1}{2} p : b = s, \alpha (AH) : q (AB)$, and $v, \alpha - q$

$(AZ - AB) = v, E - e (BZ)$, but $\frac{1}{2} E + e (az =

180^\circ - \alpha) = 90^\circ - \frac{1}{2} \alpha$; And $\frac{1}{2} E + e \mp \frac{1}{2} E - e$

$= E, e$, required.

And because of the Air's Resistance, the Line of Pro-
jects is not exactly Parabolical, but rather a kind of an
Hyperbola; which, if consider'd, and apply'd to Practice,
would render the Computation far more operose, and
the very small Difference (as Experience shews in Heavy
Shot) would, in a great Measure, lessen the Elegancy of
the Demonstrations given by Accounting for it; since
the former Rules are sufficiently exact and easily for
Practice.

The Theory of the Motion of Projects is so perplex'd a
Subject, and depends so much upon Physical Observations,
that such Accuracy cannot be expected therein, as don't
require some Allowances. But the Resistance of the Me-
dium, and other Accidental Impediments, may be in some
measure
measure Reศified, by supposing the Shot to move in a Right Line to a certain Distance (d) from the Axis of the Gun, and afterwards to describe the Curve of a Parabola.

Given, E, e; H, h; Rxq, p, d. Since d + y, and x are given (by Trigonom.) also y = √px√x (by 36. 68), and d =

\[ D + x = y = \sqrt{x + \sqrt{x}} \cdot \sqrt{y} \cdot \sqrt{y} \]

Th. \( p = \)

\[ D + y = \sqrt{x + \sqrt{x}} \cdot \sqrt{y} \cdot \sqrt{y} \]

And d = d + y - \sqrt{px (y)}

Given \( d + p, \text{Le, n} \); Reqd. b. Since r = d + y : s = ds + ys

\[ r (x + n) = y^2 + p + n, \text{th. } ry^2 - psy = pds + \]

\[ + mp; \text{ (let } r : s : p : q, \text{ or } rq = sp) \text{ and } y^2 - qy = dq + \]

\[ np, \text{ th. y is given}; \text{ And } r: d + y: c : e = b \text{ sought}. \]

Given \( d, b, y, n \); Reqd. l.e. Bec. \( x + n = (y^3 : p \]

\[ \pm n) y^2 + np \pm p, \text{ th. } y^2 + np \pm p^2 + b^2 = y + d \]

\[ \pm 2np - p^2 \times y^2 - 2dp^2 y = d^2 - b^2 - n^2 \times p^2, \text{ th. y is given}; \text{ And } d + y: r : c: \text{Le sought}. \]

84. Of Optics.

That Science, which accounts for the various Appearances of Objects, from the Reflexions, Refractions, and Inflexions of the Rays of Light, is called Optics.

Experience shews that the Light of the Sun consists of Rays that are differently Reflectible and Refrangible. The Angle comprehended between the Incident, Reflectted, or Refracted Ray, and the Perpendicular to the Reflecting, or Refracting Surface is called the Angle of Incidence, Reflexion, or Refraction. And these three Angles lie in the same Plane. The Refraction out of a Rarer Medium into a Denier is made towards the Perpendicular; and the contrary.

And, supposing Nature's Method of working to be the most easy and expedite, if there be given two Points (B, P) in Mediums of differentgiven Densities, and the Position of the Plane (b) dividing those Mediums; Reqd. the Point (e) in that Plane, thro' which the Body passes by taking the shortest Time to move from B to P.
Synopsis

Let Be, eP = s, S; Bb, Pp = c; bP = a, be = c, then pe = a - c; let v, V = Velocity in s, S; Th. since r in s:

\[t, S : sV : S v, \text{ th. } r \text{ in } s + S \text{ is } s^{1/2} + r^{1/2} \times V + \]

\[C^{1/2} - a - c^{1/2} \times v, \text{ which if a Minimum, then } V^{1/2} = s \]

\[s^{1/2} - v a^{1/2} : S = s. \text{ Th. } S v = s v \times a - c, \text{ th. } SV \]

\[s v : a - c^{1/2}, \text{ Th. (supposing } s = S) v : V = x : a - c^{1/2} \]

\[(:: m : n) \text{ i.e. The Sines of the } L \text{ of Incidence, and of the Refracted } L \text{ are directly as the Velocities, or reciprocally as the Densities of the Mediums.} \]

But if a Body moving from B to e be Reflected to E, bec. it continues in the same Medium, th. \( v = V \), consq. \( s : S = x : a - c, \text{ and } L \text{ at } b \text{ and } b \text{ are right, th. } L \text{ Beb} = L \text{ Beb, i.e. The } L \text{ of Incidence } = L \text{ of Reflexion.} \)

Homogeneal Rays, (or those of like Refrangibility) which flow from several Points of any Object, and fall almost Perpendicularly on any Reflecting, or Refracting Surface, shall afterwards Diverge, or Converge, to, or from so many other Points, or be Parallel to so many other Lines, either accurately or very near. The same happens if the Rays be reflected or refracted successively by two or more Surfaces. That Point to, or from which Rays Converge, or Diverge is called the Focus.

Given a double Convex Lens (GLy) whose Thickness is Gy (\( = t \)), and O a Point or Object in the Axis of the Lens; the Ratio of Refraction being as m to n : Reqd. the Point (F) at which the Beams that are nearest the Axis of the Lens are collected.

Let \( O G (= O P) = d \), the Distance of the Object, \( C G (= CP) = r \), and \( c y (= c \pi) = e \), the Radius of the Segment towards and from the Object: And let \( S, L \text{ Incid. (OPM)} : S, \text{ Refr. } L \text{ (NPM or CP)} : m : n \); then, in very small \( L \), \( L \text{ Incid. : Refr. } L : m : n \). Therefor. \( d : r : L C : L O \); th. \( d + r \) is as the \( L \text{ Incid. (OPM)} \) and \( m : n : d + r ; n d + r = m \), which is as the Refr. \( L \text{ (CP)} \)
Palmariorum Matheseos.

$L(C\varphi)$; But $LPCO - CP\varphi$ (or $d = nd + nr = m$) = $L\varphi'.O$. And the $L\varphi:LO:PO:P\varphi$ (or $G\varphi$)

\[ \frac{md}{md} + \frac{nd}{nd} + \frac{nr}{nr} \] (that is, the Beams from $O$ by the first Refraction will be collected in $\varphi$.)* If $nr$ be greater, or equal to $md - nd$, then the Beams after Refraction go Diverging from, or Parallel to the Axis, and the Point $\varphi$ is on the same side beyond $P$, or Infinitely distant. Also (if $\varphi = G\varphi - G\psi = \phi\varphi$ or $\eta\varphi$) $\varphi = c : \phi$; th. $\varphi + \eta$ is as the $L\ Incid.$ ($K\pi\phi$) and

\[ m : m' = \phi : \eta + m\eta + \frac{md}{nd} = n, \] which is as the Refl.($K\varphi\pi\eta$)

But $L\ K\pi\varphi\eta - K\pi\varphi$ (or $\eta\varphi + \frac{md}{nd} + \frac{nr}{n} = \eta$) = $L\eta\pi\varphi$.
And the $L\eta\pi\varphi : L\pi\varphi = \eta\pi\varphi = \eta\pi$ = $\eta\pi\varphi$

\[ \frac{md\eta}{md} + \frac{nd\eta}{nd} + \frac{nr\eta}{nr} = \eta\varphi \] (if $a = n = m - n$) unto

\[ \frac{md\eta}{md} + \frac{nd\eta}{nd} + \frac{nr\eta}{nr} = \eta\varphi \] Whence, any four of these five Quantities ($d_1, r, \varphi, a, \eta$) being given, the fifth is readily found. And,

In Diverging Rays, falling on a double Convex, or Concave (where it is $+d$, & $\pm r$, $\mp \varphi$) $+ \eta\varphi$

\[ dr + \eta\varphi \pm \eta\varphi = \pm f; \] But, in the Convex, if $\eta\varphi$

\[ dr + \eta\varphi \] then 'tis $- f$.

In Converging Rays, falling on a double Convex, or Concave, (where 'tis $- d$, and $\pm r$, $\mp \varphi$) $- \eta\varphi$

\[ dr + \eta\varphi \pm \eta\varphi = \pm f; \] But in the Concave, if $\eta\varphi$

\[ dr + \eta\varphi \] then 'tis $+ f$.

In Diverging, or Converging Rays, falling on a Meniscus, $\mp \eta\varphi$ = $\pm dr + \eta\varphi + \eta\varphi + f$, whilst $dr - \eta\varphi$ $> 0$ or $< \eta\varphi$; But if $dr - \eta\varphi < 0$, or $> \eta\varphi$, 'tis $= - f$.

In Parallel Rays, falling on a Double Convex, (where $d = \omega$) $f = a \varphi \frac{1}{r + \varphi}$ or (in Glass, where $m : n : 3$

\[ = 2\varphi \frac{1}{r + \varphi} \] Th. if $r = \varphi$, then $f = r$.

In Diverging Rays, on a Plane-Convex, (where $r = \infty$) $f = a \varphi = d - \eta\varphi$.

If $r = 2\varphi$, & $r = \varphi$, and $d = \infty$, then $f = \frac{2nr}{mr} = \frac{2m}{2n} = \frac{2m}{2n}$ (in Sphere of Glass)

\[ \frac{2m}{2n} = \frac{2m}{2n} = \frac{2m}{2n} \] or in a Sphere of Water (where $m : n : 4 : 3$) $f = r$. And
And wherever the Rays, which come from all the Points of any Object, meet again in so many Points, after they have been made to Converge by Reflexion or Refraction, there will they make a Picture of the Object. Now, a Lens being given, to find the Distance, whereas an Object being place'd, shall be Represen'ted as large as the Object itself; Here $d = f$, Th. $2 \frac{ar}{r} = d$; or if $r = \varepsilon$, then $d = ar$; or, in Glass $d = 2r$. In a Plane Convex, $d = 2ar$, i.e. in Glass, $d = 4r$. This is of singular use for Drawing things in their just Magnitude, by Transmitting the Species by a Glass into a dark Room, whereby, not only the True Figure and Shades are perfectly given, but also the Colors nearly as Vivid as the Life. 'Tis also facile from hence, to Magnify or Diminish an Object in any given Proportion. An Object seen by Reflexion or Refraction, appears in that Place from whence the Rays, after their last Reflexion or Refraction, diverge in falling on the Spectator's Eye. Whence, As the distance of the Object from the Lens is to the Diameter of the Object's Magnitude, so is the Distance of the Image to its Diameter. And if the Object be seen, thro' two or more Glasses, every Glass shall make a New Image, and the Object shall appear in the Place, and of the Bigness of the last Image: Upon this depends the Theory of Microscopes and Telescopes.

Having given $(r)$ the Rad. of one Segment of a Lens; to find the Rad. of the Convexity or Concavity, necessary to make a vastly distant Object be represented at a given Focus. Since $r, a, f,$ and $d$, are given, Th. $d\frac{r}{f} = \frac{adr}{dr} + \frac{arf}{ar} = (\text{because } d = \infty) \frac{rf}{ar} - f = \varepsilon$, or $(if f > ar) = -\varepsilon$, and $rf - f - ar = \text{Radius of the Concavity.}$

**NOTE.** That by putting $2rx - xx = \text{to the Value of } y^2$ in any Curve, and rejecting the Terms concern'd with the Powers of $x$, the Radius of a Circle Equicurve with that of the Lens, at the Vertex, is determin'd, (and in the Conic Sections will be $\frac{1}{x}$ the Parameter); this substituted in the room of $r, \varepsilon$, will give the Theorem for a Lens of any given Figure. And where the Rays are Reflected from (GL) the Concave Surface of a Speculum, (bec. 'cis + $d$, + $m$, $f$, $-r$, $-n$, and $m = n$,) $f = \frac{dr}{2d - r}$.

**FINIS.**